2 LINEAR SYSTEMS

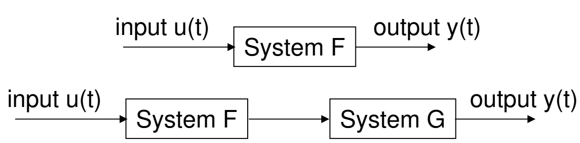
We will discuss what we mean by a linear time-invariant system, and then consider several useful transforms.

2.1 Definition of a System

In short, a system is any process or entity that has one or more well-defined inputs and one or more well-defined outputs. Examples of systems include a simple physical object obeying Newtonian mechanics, and the US economy!

Systems can be physical, or we may talk about a mathematical description of a system. The point of modeling is to capture in a mathematical representation the behavior of a physical system. As we will see, such representation lends itself to analysis and design, and certain restrictions such as linearity and time-invariance open a huge set of available tools.

We often use a block diagram form to describe systems, and in particular their interconnections:



In the second case shown, $y(t) = G[F[u(t)]]$.

Looking at structure now and starting with the most abstracted and general case, we may write a system as a function relating the input to the output; as written these are both functions of time:

$$y(t) = F[u(t)]$$

The system captured in F can be a multiplication by some constant factor - an example of a static system, or a hopelessly complex set of differential equations - an example of a dynamic system. If we are talking about a dynamical system, then by definition the mapping from $u(t)$ to $y(t)$ is such that the current value of the output $y(t)$ depends on the *past history* of $u(t)$. Several examples are:

$$\begin{aligned} y(t) &= \int_{t-3}^{t} u^2(t_1)dt_1, \\ y(t) &= u(t) + \sum_{n=1}^{N} u(t - n\delta t). \end{aligned}$$

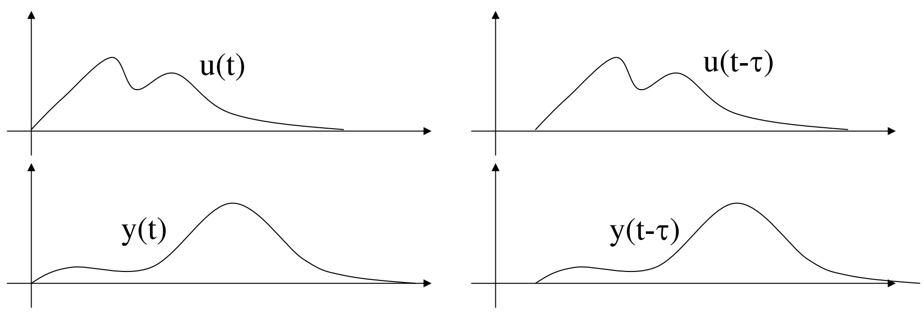
In the second case, δt is a constant time step, and hence $y(t)$ has embedded in it the current input plus a set of N delayed versions of the input.

2.2 Time-Invariant Systems

A dynamic system is time-invariant if shifting the input on the time axis leads to an equivalent shifting of the output along the time axis, with no other changes. In other words, a time-invariant system maps a given input trajectory $u(t)$ no matter when it occurs:

$$y(t-\tau) = F[u(t-\tau)].$$

The formula above says specifically that if an input signal is delayed by some amount τ, so will be the output, and with no other changes.



An example of a physical time-varying system is the pitch response of a rocket, $y(t)$, when the thrusters are being steered by an angle $u(t)$. You can see first that this is an inverted pendulum problem, and unstable without a closed-loop controller. It is time-varying because as the rocket burns fuel its mass is changing, and so the pitch responds differently to various inputs throughout its flight. In this case the "absolute time" coordinate is the time since liftoff.

To assess whether a system is time-varying or not, follow these steps: replace $u(t)$ with $u(t-\tau)$ on one side of the equation, replace $y(t)$ with $y(t-\tau)$ on the other side of the equation, and then check if they are equal. Here are several examples.

$$y(t) = u(t)^{3/2}$$

This system is clearly time-invariant, because it is a static map. Next example:

$$y(t) = \int_0^t \sqrt{u(t_1)}dt_1$$

Replace $u(t_1)$ with $u(t_1-\tau)$ in the right hand side and carry it through:

$$\int_0^t \sqrt{u(t_1-\tau)}dt_1 = \int_{-\tau}^{t-\tau} \sqrt{u(t_2)}dt_2.$$

The left hand side is simply

$$y(t-\tau) = \int_0^{t-\tau} \sqrt{u(t_1)}dt_1$$

Clearly the right and left hand sides are not equal (the limits of integration are different), and hence the system is not time-invariant. As another example, consider

$$y(t) = \int_{t-5}^{t} u^2(t_1)dt_1$$

The right-hand side becomes with the time shift

$$\int_{t-5}^{t} u^2(t_1 - \tau)dt_1 = \int_{t-5-\tau}^{t-\tau} u^2(t_2)dt_2,$$

whereas the left-hand side is

$$y(t-\tau) = \int_{t-5-\tau}^{t-\tau} u^2(t_1)dt_1;$$

the two sides of the defining equation are equal under a time shift τ, and so this system is time-invariant.

A subtlety here is encountered when considering inputs that are zero before time zero - this is the usual assumption in our work, namely $u(t) = 0$ for $t \leq 0$. While linearity is not affected by this condition, time invariance is, because the assumption is inconsistent with *advancing* a signal in time. Clearly part of the input would be truncated! Restricting our discussion to signal *delays* (the insertion of $-\tau$ into the argument, where strictly $\tau > 0$) resolves the issue, and preserves time invariance as needed.

2.3 Linear Systems

Next we consider linearity. Roughly speaking, a system is linear if its behavior is scale-independent; a result of this is the superposition principle. More precisely, suppose that $y_1(t) = F[u_1(t)]$ and $y_2(t) = F[u_2(t)]$. Then linearity means that for any two constants α_1 and α_2,

$$y(t) = \alpha_1 y_1(t) + \alpha_2 y_2(t) = F[\alpha_1 u_1(t) + \alpha_2 u_2(t)].$$

A simple special case is seen by setting $\alpha_2 = 0$:

$$y(t) = \alpha_1 y_1(t) = F[\alpha_1 u_1(t)],$$

making clear the scale-invariance. If the input is scaled by α_1, then so is the output. Here are some examples of linear and nonlinear systems:

$$\begin{aligned}
y(t) &= c\frac{du}{dt} \text{ (linear and time-invariant)} \\
y(t) &= \int_0^t u(t_1)dt_1 \text{ (linear but not time-invariant)} \\
y(t) &= 2u^2(t) \text{ (nonlinear but time-invariant)} \\
y(t) &= 6u(t) \text{ (linear and time-invariant).}
\end{aligned}$$

Linear, time-invariant (LTI) systems are of special interest because of the powerful tools we can apply to them. Systems described by sets of linear, ordinary or differential differential equations having constant coefficients are LTI. This is a large class! Very useful examples include a mass m on a spring k, being driven by a force $u(t)$:

$$my''(t) + ky(t) = u(t),$$

where the output $y(t)$ is interpreted as a position. A classic case of an LTI partial differential equation is transmission of lateral waves down a half-infinite string. Let m be the mass per unit length, and T be the tension (constant on the length). If the motion of the end is $u(t)$, then the lateral motion satisfies

$$m\frac{\partial^2 y(t,x)}{\partial t^2} = T\frac{\partial^2 y(t,x)}{\partial x^2}$$

with $y(t, x = 0) = u(t)$. Note that the system output y is not only a function of time but also of space in this case.

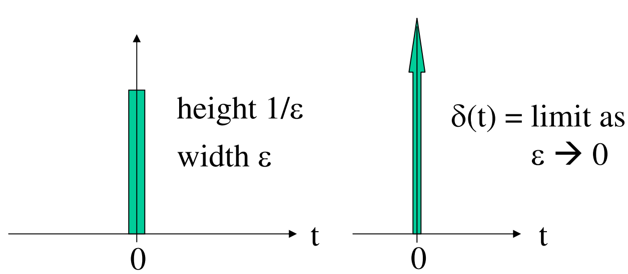
2.4 The Impulse Response and Convolution

A fundamental property of LTI systems is that they obey the convolution operator. This operator is defined by

$$y(t) = \int_{-\infty}^{\infty} u(t_1)h(t - t_1)dt_1 = \int_{-\infty}^{\infty} u(t - t_1)h(t_1)dt_1.$$

The function $h(t)$ above is a particular characterization of the LTI system known as the *impulse response* (see below). The equality between the two integrals should be clear since the limits of integration are infinite. The presence of the t_1 and the $-t_1$ term inside the integrations tells you that we have integrals of products - but that one of the signals is turned around. We will describe the meaning of the convolution more fully below.

To understand the impulse response, first we need the concept of the impulse itself, also known as the delta function $\delta(t)$. Think of a rectangular box centered at time zero, of width (time duration) ϵ, and height (magnitude) $1/\epsilon$; the limit as $\epsilon \longrightarrow 0$ is the δ function. The area is clearly one in any case.



The inner product of the delta function with any function is the value of the function at zero time:

$$\int_{-\infty}^{\infty} f(t)\delta(t)dt = \int_{-\epsilon/2}^{\epsilon/2} f(t)\delta(t)dt = f(t=0)\int_{-\epsilon/2}^{\epsilon/2} \delta(t)dt = f(0).$$

More usefully, the delta function can pick out the function value at a given, nonzero time ξ:

$$\int_{-\infty}^{\infty} f(t)\delta(t-\xi)dt = f(\xi).$$

Returning now to the impulse response function $h(t)$, it is, quite simply, the output of the LTI system, when driven by the delta function as input, that is $u(t) = \delta(t)$, or $h(t) = F[\delta(t)]$. In practical terms, we can liken $h(t)$ to the response of a mechanical system when it is struck very hard by a hammer!

Next we put the delta function and the convolution definition together, to show explicitly that the response of a system to arbitrary input $u(t)$ is the convolution of the input and the impulse response $h(t)$. This is what is stated in the definition given at the beginning of this section. First, we note that

$$\begin{aligned} u(t) &= \int_{-\infty}^{\infty} u(\xi)\delta(\xi - t)d\xi \\ &= \int_{-\infty}^{\infty} u(\xi)\delta(t-\xi)d\xi \text{ (because the impulse is symmetric about zero time).} \end{aligned}$$

Now set the system response $y(t) = F[u(t)]$, where F is an LTI system - we will use its two properties below.

$$\begin{aligned} y(t) &= F\left[\int_{-\infty}^{\infty} u(\xi)\delta(t-\xi)d\xi\right] \\ &= \int_{-\infty}^{\infty} u(\xi)F[\delta(t-\xi)]d\xi \text{ (using linearity)} \\ &= \int_{-\infty}^{\infty} u(\xi)h(t-\xi)d\xi \text{ (using time invariance),} \end{aligned}$$

and this indeed is the definition of convolution, often written as $y(t) = h(t) * u(t)$.

An intuitive understanding of convolution can be gained by thinking of the input as an infinite number of scaled delta functions, placed very closely together on the time axis. Explaining the case with the integrand $u(t-\xi)h(\xi)$, we see the convolution integral will call up all these virtual impulses, referenced to time t, and multiply them by the properly shifted impulse responses. Consider one impulse only that occurs at time $t = 2$, and we are interested in the response at $t = 5$. Then $u(t) = \delta(t-2)$ or $u(t-\xi) = \delta(t-2-\xi)$. The integrand will thus be nonzero only when $t-2-\xi$ is zero, or $\xi = t-2$. Now $h(\xi) = h(t-2)$ will be $h(3)$ when $t = 5$, and hence it provides the impulse response three time units after the impulse occurs, which is just what we wanted.

2.5 Causal Systems

All physical systems respond to input only *after* the input is applied. In math terms, this means $h(t) = 0$ for all $t < 0$. For convenience, we also usually consider input signals to be zero before time zero. The convolution is adapted in a very reasonable way:

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} u(\xi)h(t-\xi)d\xi \\ &= \int_{0}^{\infty} u(\xi)h(t-\xi)d\xi \\ &= \int_{0}^{t} u(\xi)h(t-\xi)d\xi. \end{aligned}$$

The lower integration limit is set by the assumption that $u(t) = 0$ for $t < 0$, and the upper limit is set by the causality of the impulse response. The complementary form with integrand $u(t-\xi)h(\xi)$ also holds.

2.6 An Example of Finding the Impulse Response

Let's consider the differential equation $mx''(t) + bx'(t) + cx(t) = \delta(t)$, with the initial conditions of $x(0) = x'(0) = 0$. We have

$$\begin{aligned} \int_{-\epsilon/2}^{\epsilon/2} [mx'' + bx' + cx]dt &= \int_{-\epsilon/2}^{\epsilon/2} \delta(t)dt = 1, \text{ so that} \\ m(x'(0^+) - x'(0^-)) &= 1 \end{aligned}$$

The + superscript indicates the instant just after zero time, and the − superscript indicates the instant just before zero time. The given relation follows because at time zero the velocity and position are zero, so it must be the acceleration which is very large. Now since $x'(0-) = 0$, we have $x'(0+) = 1/m$. This is very useful - the initial velocity after the mass is hit with a $\delta(t)$ input. In fact, this *replaces* our previous initial condition $x'(0) = 0$, and we can treat the differential equation as homogeneous from here on. With $x(t) = c_1 e^{s_1 t} + c_2 e^{s_2 t}$, the governing equation becomes $ms_i^2 + bs_i + k = 0$ so that

$$s = -\frac{b}{2m} \pm \frac{\sqrt{b^2 - 4km}}{2m}.$$

Let $\sigma = b/2m$ and

$$\omega_d = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}},$$

and assuming that $b^2 < 4km$, we find

$$h(t) = \frac{1}{m\omega_d} e^{-\sigma t} sin(\omega_d t), \quad t \geq 0.$$

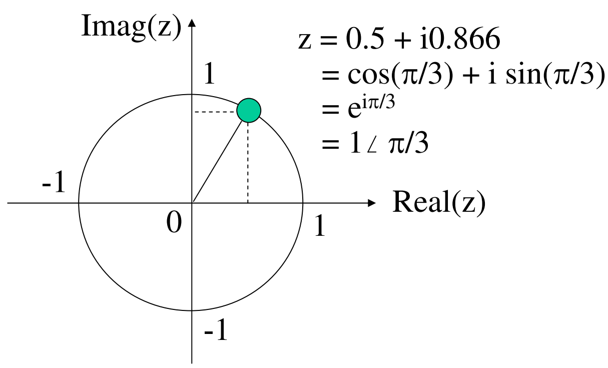
As noted above, once the impulse response is known for an LTI system, responses to all inputs can be found:

$$x(t) = \int_0^t u(\tau)h(t-\tau)d\tau.$$

In the case of LTI systems, the impulse response is a complete definition of the system, in the same way that a differential equation is, with zero initial conditions.

2.7 Complex Numbers

The complex number $z = x + iy$ is interpreted as follows: the real part is x, the imaginary part is y, and $i = \sqrt{-1}$ (imaginary). DeMoivre's theorem connects complex z with the complex exponential. It states that $\cos\theta + i\sin\theta = e^{i\theta}$, and so we can visualize any complex number in the two-plane, where the axes are the real part and the imaginary part. We say that $Re\left(e^{i\theta}\right) = cos\theta$, and $Im\left(e^{i\theta}\right) = sin\theta$, to denote the real and imaginary parts of a complex exponential. More generally, $Re(z) = x$ and $Im(z) = y$.



A complex number has a magnitude and an angle: $|z| = \sqrt{x^2+y^2}$, and $\arg(z) = atan2(y,x)$. We can refer to the $[x,y]$ description of z as Cartesian coordinates, whereas the [magnitude, angle] description is called polar coordinates. This latter is usually written as $z = |z|\angle\arg(z)$. Arithmetic rules for two complex numbers z_1 and z_2 are as follows:

$$\begin{aligned}
z_1 + z_2 &= (x_1 + x_2) + i(y_1 + y_2) \\
z_1 - z_2 &= (x_1 - x_2) + i(y_1 - y_2) \\
z_1 \cdot z_2 &= |z_1||z_2|\angle\arg(z_1) + \arg(z_2) \\
z_1/z_2 &= \frac{|z_1|}{|z_2|}\angle\arg(z_1) - \arg(z_2)
\end{aligned}$$

Note that, as given, addition and subtraction are most naturally expressed in Cartesian coordinates, and multiplication and division are cleaner in polar coordinates.

2.8 Fourier Transform

The Fourier transform is the underlying principle for frequency-domain description of signals. We begin with the Fourier series.

Consider a signal $f(t)$ continuous on the time interval $[0, T]$, which then repeats with period T off to negative and positive infinity. It can be shown that

$$\begin{aligned}
f(t) &= A_o + \sum_{n=1}^{\infty} [A_n \cos(n\omega_o t) + B_n \sin(n\omega_o t)], \text{ where} \\
\omega_o &= 2\pi/T, \\
A_0 &= \frac{1}{T}\int_0^T f(t)dt, \\
A_n &= \frac{2}{T}\int_0^T f(t)\cos(n\omega_o t)dt, \text{ and} \\
B_n &= \frac{2}{T}\int_0^T f(t)\sin(n\omega_o t)dt.
\end{aligned}$$

This says that the time-domain signal $f(t)$ has an exact (if you carry all the infinity of terms) representation of a constant plus scaled cosines and sines. As we will see later, the impact of this second, frequency-domain representation is profound, as it allows an entirely new set of tools for manipulation and analysis of signals and systems. A compact form of these expressions for the Fourier series can be written using complex exponentials:

$$\begin{aligned}
f(t) &= \sum_{n=-\infty}^{\infty} C_n e^{in\omega_o t}, \text{ where} \\
C_n &= \frac{1}{T}\int_0^T f(t)e^{-in\omega_o t}dt.
\end{aligned}$$

Of course, C_n can be a complex number.

In making these inner product calculations, orthogonality of the harmonic functions is useful:

$$\begin{aligned}
\int_0^{2\pi} \sin nt \ \sin mt \ dt &= 0, \text{ for } n \geq 1, \ m \geq 1, \ n \neq m \\
\int_0^{2\pi} \cos nt \ \cos mt \ dt &= 0, \text{ for } n \geq 1, \ m \geq 1, \ n \neq m \\
\int_0^{2\pi} \sin nt \ \cos mt \ dt &= 0, \text{ for } n \geq 1, \ m \geq 1.
\end{aligned}$$

Now let's go to a different class of signal, one that is not periodic, but has a finite integral of absolute value. Obviously, such a signal has to approach zero at distances far from the origin. We can write a more elegant transformation:

$$\begin{aligned}
F(\omega) &= \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt, \\
f(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega\tau}d\tau.
\end{aligned}$$

This is the real Fourier transform: a time-domain signal is transformed into a (complex) frequency-domain version, and it can be transformed back. On working it through, we see that derivatives and integrals look this way through the transform:

$$\begin{aligned} f(t) &\longleftrightarrow F(\omega) \\ \frac{d^n f(t)}{dt^n} &\longleftrightarrow (i\omega)^n F(w) \\ \int_{-\infty}^{t} f(\tau)d\tau &\longleftrightarrow \frac{1}{i\omega}F(w). \end{aligned}$$

Another very important property of the Fourier transform is Parseval's Relation:

$$\int_{-\infty}^{\infty} f^2(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)F^*(\omega)d\omega = \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega,$$

where the $*$-superscript indicates the complex conjugate. We will give more properties for the related Laplace transform in a later section. But as is, the Fourier transform is immediately useful for solving linear differential equations with constant coefficients (LTI systems):

$$\begin{aligned} mx'' + bx' + cx = u(t) \quad \longleftrightarrow \quad & [-m\omega^2 + i\omega b + k]X(\omega) = U(\omega), \text{ so that} \\ & X(\omega) = \frac{1}{-m\omega^2 + i\omega b + k}U(\omega). \end{aligned}$$

Hence, the action of the differential equation to relate $f(t)$ with $x(t)$ is, in the frequency domain, captured by the function

$$H(\omega) = \frac{1}{-m\omega^2 + i\omega b + k}$$

Putting two and two together, we then assert that $X(\omega) = H(\omega)U(\omega)$; in the Fourier space, the system response is the product of impulse response function, and the input! To back this up, we show now that convolution in the time-domain is equivalent to multiplication in the frequency domain:

$$\begin{aligned} X(\omega) &= \mathcal{F}\left[\int_{-\infty}^{\infty} u(\tau)h(t-\tau)d\tau\right] \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(\tau)h(t-\tau)e^{-i\omega t}dtd\tau \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(\tau)h(\xi)e^{-i\omega(\xi+\tau)}d\xi d\tau \quad \text{because } e^{-i\omega t} = e^{-i\omega(t-\tau+\tau)} \\ &= \int_{-\infty}^{\infty} e^{-i\omega\xi}h(\xi)d\xi \int_{-\infty}^{\infty} e^{-i\omega\tau}u(\tau)d\tau \\ &= H(\omega)U(\omega). \end{aligned}$$

The central role of the impulse response should be reiterated here. It is a complete definition of the system, and for systems of differential equations, it is a specific function of the parameters and of the frequency ω. The Fourier Transform of the impulse response called the system *transfer function*, and we often refer to the transfer function as "the system," even though it is actually a (transformed) signal.

By way of summary, we can write

$$\begin{aligned} y(t) &= h(t) * u(t), \text{ and} \\ Y(\omega) &= H(\omega)U(\omega). \end{aligned}$$

2.9 The Angle of a Transfer Function

A particularly useful property of the Fourier (and Laplace) transform is that the magnitude of the transfer function scales a sinusoidal input, and the angle of the transfer function adds to the angle of the sinusoidal input. In other words,

$$\begin{aligned} u(t) &= u_o \cos(\omega_o t + \psi) \longrightarrow \\ y(t) &= u_o |H(\omega_o)| cos(\omega_o t + \psi + \arg(H(\omega_o)). \end{aligned}$$

To prove the above relations, we'll use the complex exponential:

$$\begin{aligned} u(t) &= Re\left(u_o e^{i(\omega_o t + \psi)}\right) \\ &= Re\left(\tilde{u}_o e^{i\omega_o t}\right), \text{ making } u_o e^{i\psi} = \tilde{u}_o \text{ complex; then} \\ y(t) &= h(t) * u(t) \\ &= \int_{-\infty}^{\infty} h(\tau) u(t-\tau) d\tau \\ &= \int_{-\infty}^{\infty} h(\tau) Re\left(\tilde{u}_o e^{i\omega_o (t-\tau)}\right) d\tau \\ &= Re\left(\int_{-\infty}^{\infty} h(\tau) e^{-i\omega_o \tau} d\tau \;\; \tilde{u}_o e^{i\omega_o t}\right) \\ &= Re\left(H(\omega_o) u_o e^{i(\omega_o t + \psi)}\right) \\ &= u_o |H(\omega_o)| \cos(\omega_o t + \psi + \arg(H(\omega_o))). \end{aligned}$$

As an example, let $u(t) = 4\cos(3t + \pi/4)$, and $H(\omega) = 2i\omega/5$. Then $H(\omega_o) = H(3) = 6i/5 = 1.2\angle\pi/2$. Thus, $y(t) = 4.8\cos(3t + 3\pi/4)$.

2.10 The Laplace Transform

The causal version of the Fourier transform is the Laplace transform; the integral over time includes only positive values and hence only deals with causal impulse response functions. In our discussion, the Laplace transform is chiefly used in control system analysis and design.

2.10.1 Definition

The Laplace transform projects time-domain signals into a complex frequency-domain equivalent. The signal $y(t)$ has transform $Y(s)$ defined as follows:

$$Y(s) = L(y(t)) = \int_0^\infty y(\tau)e^{-s\tau}d\tau,$$

where s is a complex variable, properly constrained within a region so that the integral converges. $Y(s)$ is a complex function as a result. Note that the Laplace transform is linear, and so it is distributive: $L(x(t) + y(t)) = L(x(t)) + L(y(t))$. The following table gives a list of some useful transform pairs and other properties, for reference.

The last two properties are of special importance: for control system design, the differentiation of a signal is equivalent to multiplication of its Laplace transform by s; integration of a signal is equivalent to division by s. The other terms that arise will cancel if $y(0) = 0$, or if $y(0)$ is finite.

2.10.2 Convergence

We note first that the value of s affects the convergence of the integral. For instance, if $y(t) = e^t$, then the integral converges only for $Re(s) > 1$, since the integrand is e^{1-s} in this case. Although the integral converges within a well-defined region in the complex plane, the function $Y(s)$ is defined for all s through analytic continuation. This result from complex analysis holds that if two complex functions are equal on some arc (or line) in the complex plane, then they are equivalent everywhere. It should be noted however, that the Laplace transform is defined only within the region of convergence.

2.10.3 Convolution Theorem

One of the main points of the Laplace transform is the ease of dealing with dynamic systems. As with the Fourier transform, the convolution of two signals in the time domain corresponds with the multiplication of signals in the frequency domain. Consider a system whose impulse response is $g(t)$, being driven by an input signal $x(t)$; the output is $y(t) = g(t) * x(t)$. The *Convolution Theorem* is

$$y(t) = \int_0^t g(t-\tau)x(\tau)d\tau \Longleftrightarrow Y(s) = G(s)X(s).$$

Here's the proof given by Siebert:

$$
\begin{array}{rrcl}
 & y(t) & \longleftrightarrow & Y(s) \\
\text{(Impulse)} & \delta(t) & \longleftrightarrow & 1 \\
\text{(Unit Step)} & 1(t) & \longleftrightarrow & \dfrac{1}{s} \\
\text{(Unit Ramp)} & t & \longleftrightarrow & \dfrac{1}{s^2} \\
 & e^{-\alpha t} & \longleftrightarrow & \dfrac{1}{s+\alpha} \\
 & \sin \omega t & \longleftrightarrow & \dfrac{\omega}{s^2+\omega^2} \\
 & \cos \omega t & \longleftrightarrow & \dfrac{s}{s^2+\omega^2} \\
 & e^{-\alpha t}\sin \omega t & \longleftrightarrow & \dfrac{\omega}{(s+\alpha)^2+\omega^2} \\
 & e^{-\alpha t}\cos \omega t & \longleftrightarrow & \dfrac{s+\alpha}{(s+\alpha)^2+\omega^2} \\
 & \dfrac{1}{b-a}(e^{-at}-e^{-bt}) & \longleftrightarrow & \dfrac{1}{(s+a)(s+b)} \\
 & \dfrac{1}{ab}\left[1+\dfrac{1}{a-b}(be^{-at}-ae^{-bt})\right] & \longleftrightarrow & \dfrac{1}{s(s+a)(s+b)} \\
 & \dfrac{\omega_n}{\sqrt{1-\zeta^2}}\; e^{-\zeta\omega_n t}\sin \omega_n\sqrt{1-\zeta^2}t & \longleftrightarrow & \dfrac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} \\
 & 1-\dfrac{1}{\sqrt{1-\zeta^2}}\; e^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1-\zeta^2}t+\phi\right) & \longleftrightarrow & \dfrac{\omega_n^2}{s(s^2+2\zeta\omega_n s+\omega_n^2)} \\
 & \left(\phi=\tan^{-1}\dfrac{\sqrt{1-\zeta^2}}{\zeta}\right) & & \\
\text{(Pure Delay)} & y(t-\tau)1(t-\tau) & \longleftrightarrow & Y(s)e^{-s\tau} \\
\text{(Time Derivative)} & \dfrac{dy(t)}{dt} & \longleftrightarrow & sY(s)-y(0) \\
\text{(Time Integral)} & \displaystyle\int_0^t y(\tau)d\tau & \longleftrightarrow & \dfrac{Y(s)}{s}+\dfrac{\int_{0-}^{0+} y(t)dt}{s}
\end{array}
$$

$$
\begin{aligned}
Y(s) &= \int_0^\infty y(t)e^{-st}dt \\
&= \int_0^\infty \left[\int_0^t g(t-\tau)\, x(\tau)\, d\tau\right] e^{-st}\, dt \\
&= \int_0^\infty \left[\int_0^\infty g(t-\tau)\, h(t-\tau)\, x(\tau)\, d\tau\right] e^{-st}dt \\
&= \int_0^\infty x(\tau) \left[\int_0^\infty g(t-\tau)\, h(t-\tau)\, e^{-st}\, dt\right] d\tau \\
&= \int_0^\infty x(\tau)\, G(s)e^{-s\tau}\, d\tau \\
&= G(s)X(s),
\end{aligned}
$$

where $h(t)$ is the unit step function. When $g(t)$ is the impulse response of a dynamic system, then $y(t)$ represents the output of this system when it is driven by the external signal $x(t)$.

2.10.4 Solution of Differential Equations by Laplace Transform

The Convolution Theorem allows one to solve (linear time-invariant) differential equations in the following way:

1. Transform the system impulse response $g(t)$ into $G(s)$, and the input signal $x(t)$ into $X(s)$, using the transform pairs.
2. Perform the multiplication in the Laplace domain to find $Y(s)$.
3. Ignoring the effects of pure time delays, break $Y(s)$ into partial fractions with no powers of s greater than 2 in the denominator.
4. Generate the time-domain response from the simple transform pairs. Apply time delay as necessary.

Specific examples of this procedure are given in a later section on transfer functions.