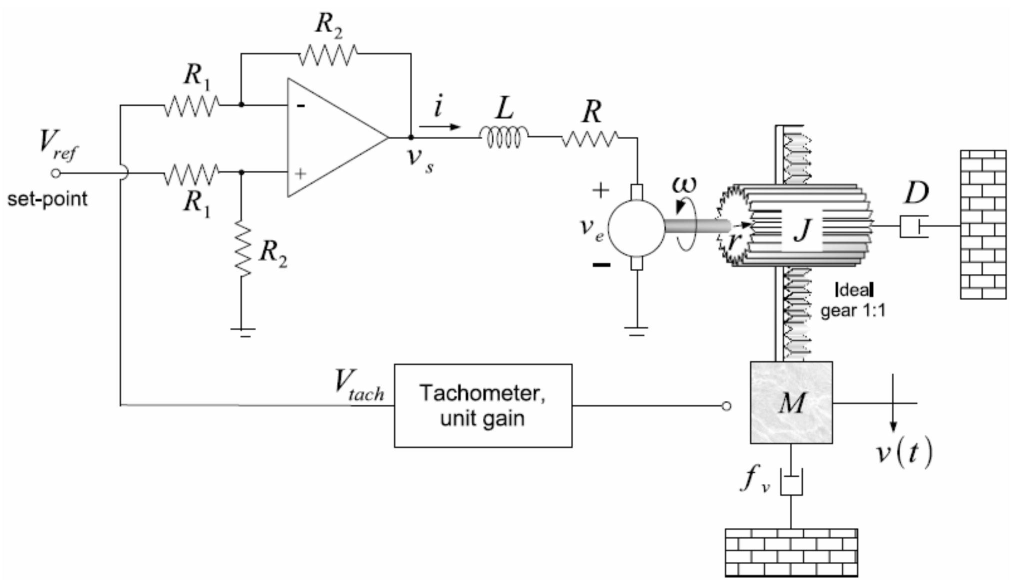
Last week: analysis of pinion-rack w velocity feedback



Calculation of the steady–state error

Transfer function: $\dfrac{V(s)}{V_{ref}(s)} = \dfrac{0.3162K}{s+2+0.3162K}$

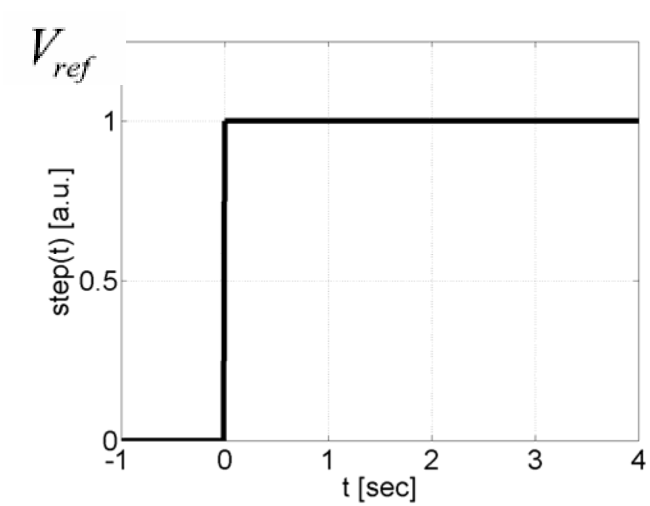
Step input: $V_{ref}(s) = \dfrac{1}{s}$

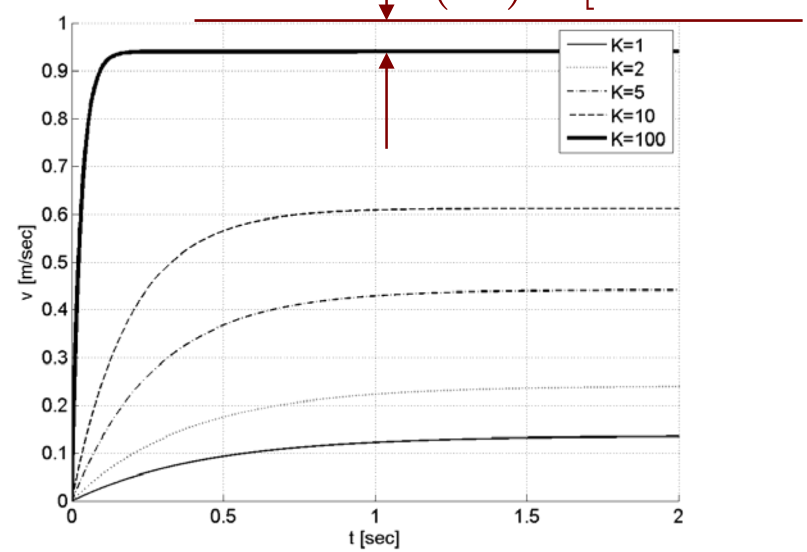
Output: $V(s) = \dfrac{1}{s} \times \dfrac{0.3162K}{s+2+0.3162K}$

Steady–state: $v(\infty) = \lim_{s\to 0} sV(s) = \dfrac{0.3162K}{2+0.3162K}$

Steady–state error: $e(\infty) = 1 - v(\infty) = \dfrac{2}{2+0.3162K}$.

$e(\infty)$ [for $K = 100$]

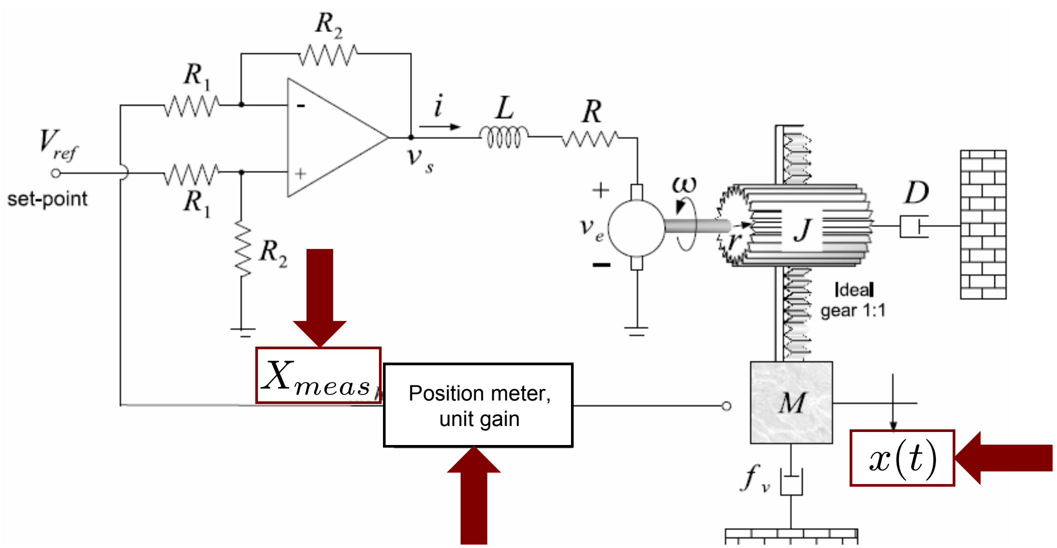




Today

- Analysis of steady-state errors
  - Definition for step, ramp, parabola inputs
  - Steady-state error in unity feedback systems
  - System type and static error constants
  - The role of integrators
  - Steady-state error in the presence of disturbances
  - Controller gain and step disturbance cancellation
- **Wednesday & Friday:** Root locus

Inserting an integrator



We've changed our mind!
Now the output is the rack **position** $x(t)$.

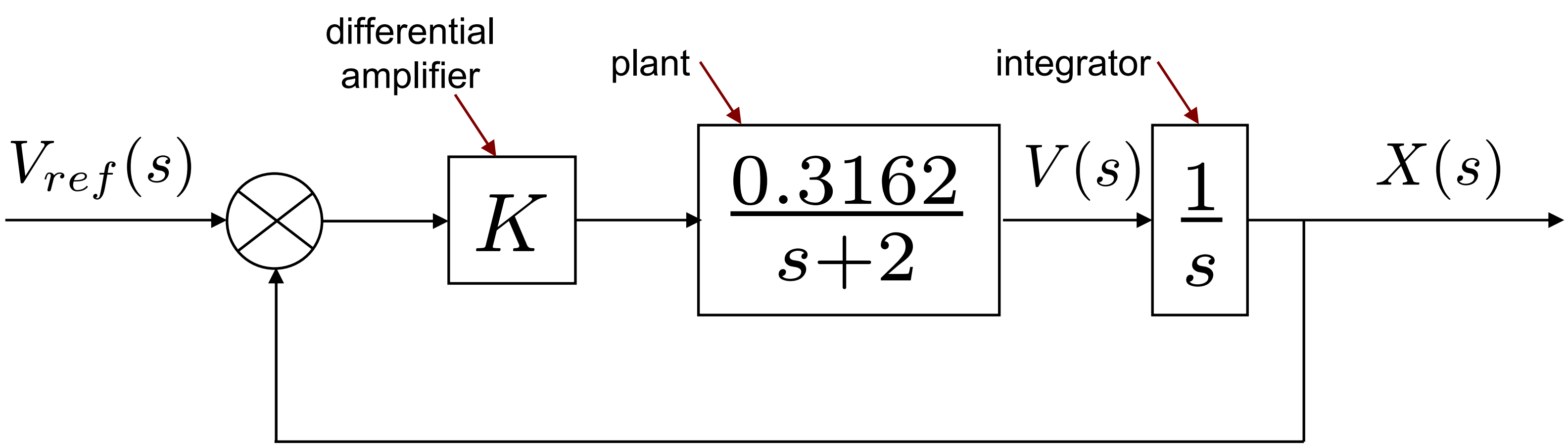
The feedback to the diff–amp must also change to measure position

Since $v(t) = \dfrac{\mathrm{d}x(t)}{\mathrm{d}t} \Leftrightarrow x(t) = \displaystyle\int_0^t v(t')\mathrm{d}t'$

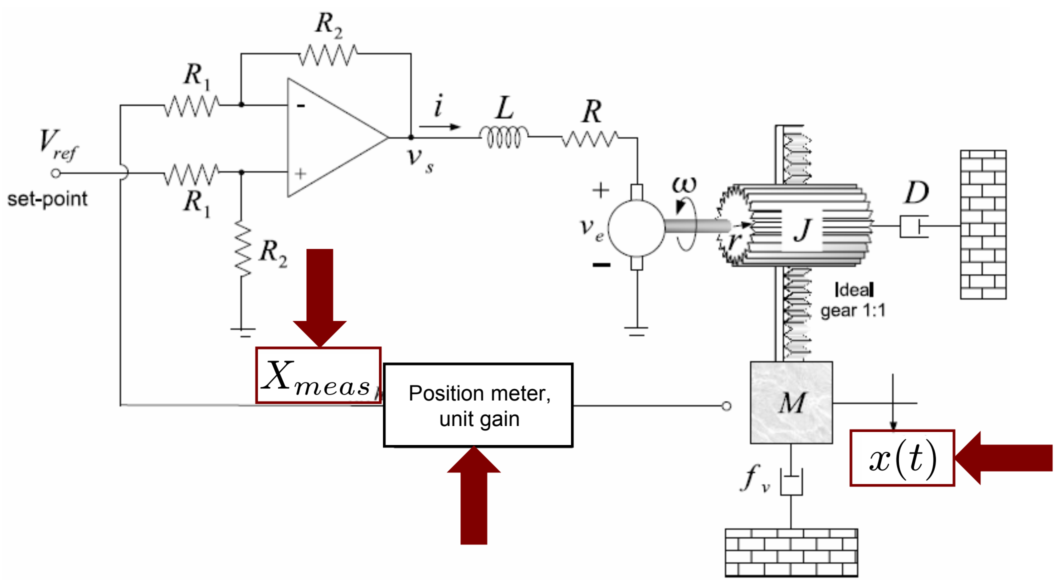
we don't need to re–compute the plant TF.

$$X(s) = \frac{1}{s} V(s),$$

where $V(s)$ is the output velocity of the previous system that we have already analyzed.



Inserting an integrator



Calculation of the closed–loop TF

Plant TF: $$\frac{X(s)}{V_s(s)} = \frac{0.3162}{s(s+2)}$$

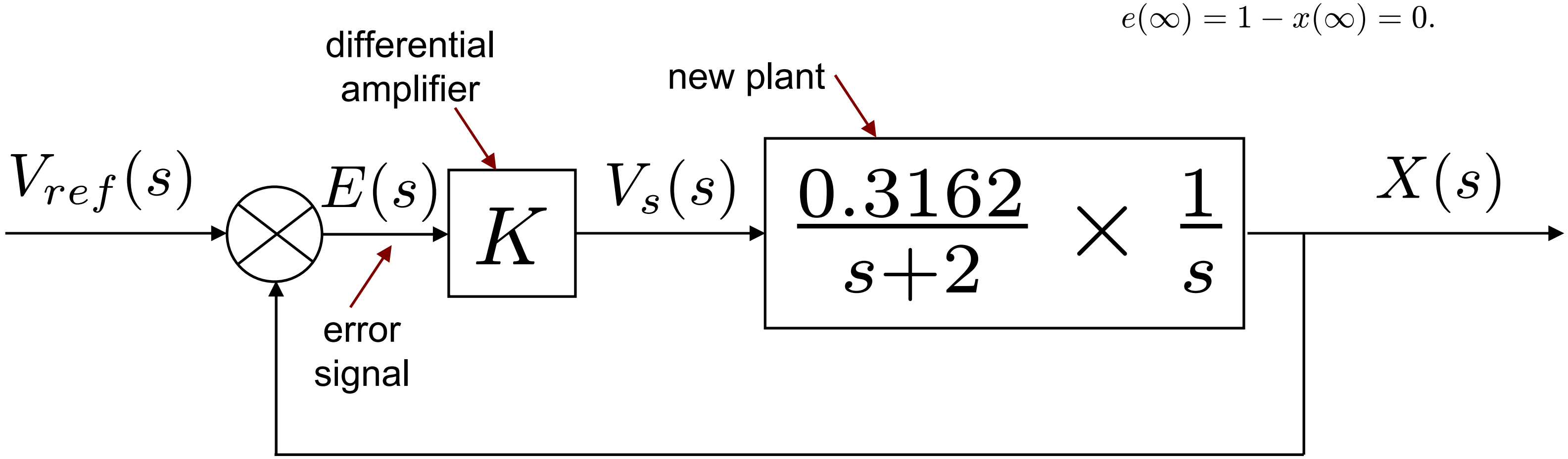
Closed loop TF: $$\frac{X(s)}{V_{ref}(s)} = \frac{0.3162K}{s^2+2s+0.3162K}$$

Calculation of the steady–state error

Step input: $$V_{ref}(s) = \frac{1}{s}$$

$$x(\infty) = \lim_{s\to 0} s \times \frac{1}{s} \times \frac{0.3162K}{s^2+2s+0.3162K} = 1.$$

$$e(\infty) = 1 - x(\infty) = 0.$$



Generalizing: different system inputs

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Please see: Table 7.1 and Fig. 7.1 in Nise, Norman S. *Control Systems Engineering*. 4th ed. Hoboken, NJ: John Wiley, 2004.

Generalizing: steady-state error for arbitrary input

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Please see Fig. 7.2 in Nise, Norman S. *Control Systems Engineering*. 4th ed. Hoboken, NJ: John Wiley, 2004.

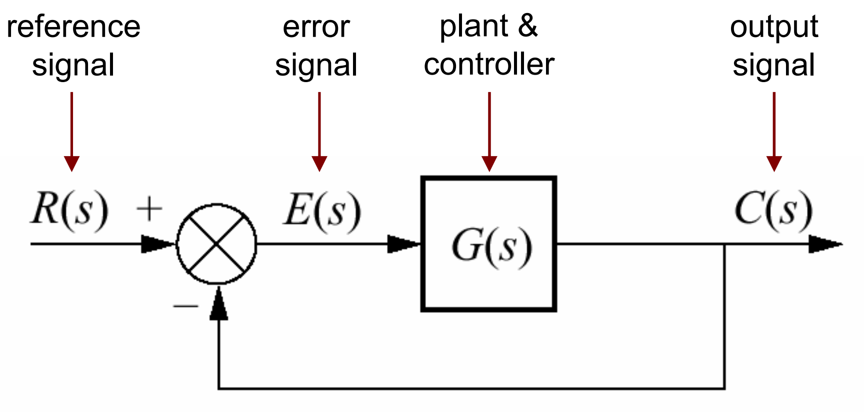
- Unit step input:
  Steady-state error =
  unit step – output as t→∞
- Ramp input:
  Steady-state error =
  ramp – output as t→∞

Generally, the steady–state error is defined as

$$e(\infty) = \lim_{t\to\infty} \left[r(t) - c(t) \right] = \lim_{s\to 0} s \left[R(s) - C(s) \right],$$

where the last equality follows from the final value theorem.

Generalizing: steady-state error for arbitrary system, unity feedback



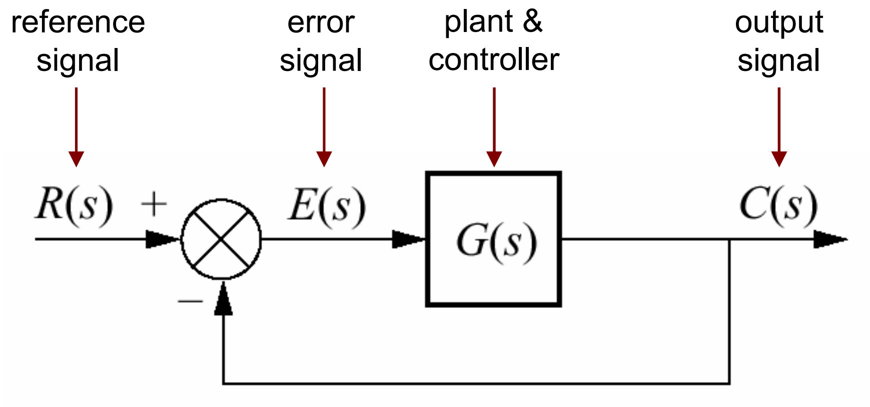
From the definition of the steady–state error,

$$e(\infty) = \lim_{s\to 0} s\left[R(s) - C(s)\right] = \lim_{s\to 0} sE(s).$$

From the block diagram we can also see that

$$\frac{C(s)}{E(s)} = G(s) \Rightarrow E(s) = \frac{C(s)}{G(s)}.$$

Generalizing: steady-state error for arbitrary system, unity feedback



Recall the closed–loop TF of the unity feedback system

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} \Rightarrow C(s) = \frac{R(s)G(s)}{1+G(s)}.$$

Substituting into the two formulae from the previous page,

$$E(s) = \frac{R(s)}{1+G(s)} \Rightarrow e(\infty) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{sR(s)}{1+G(s)}.$$

Steady-state error and static error constants

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Please see: Table 7.1 in Nise, Norman S. *Control Systems Engineering*. 4th ed. Hoboken, NJ: John Wiley, 2004.

$$e(\infty) = \lim_{s\to 0} \frac{sR(s)}{1+G(s)}.$$

$$\begin{aligned} e(\infty) &= \lim_{s\to 0} \tfrac{1}{s} \times \tfrac{s}{1+G(s)} \\ &= \lim_{s\to 0} \tfrac{1}{1+G(s)} \\ &= \tfrac{1}{1+\lim_{s\to 0} G(s)} \equiv \tfrac{1}{1+K_p} \\ \text{where} \quad K_p &= \lim_{s\to 0} G(s). \end{aligned}$$

$$\begin{aligned} e(\infty) &= \lim_{s\to 0} \tfrac{1}{s^2} \times \tfrac{s}{1+G(s)} \\ &= \lim_{s\to 0} \tfrac{1}{s+sG(s)} \\ &= \tfrac{1}{\lim_{s\to 0} sG(s)} \equiv \tfrac{1}{1+K_v} \\ \text{where} \quad K_v &= \lim_{s\to 0} sG(s). \end{aligned}$$

$$\begin{aligned} e(\infty) &= \lim_{s\to 0} \tfrac{1}{s^3} \times \tfrac{s}{1+G(s)} \\ &= \lim_{s\to 0} \tfrac{1}{s^2+s^2G(s)} \\ &= \tfrac{1}{\lim_{s\to 0} s^2G(s)} \equiv \tfrac{1}{1+K_a} \\ \text{where} \quad K_a &= \lim_{s\to 0} s^2G(s). \end{aligned}$$

Note: the system must be **stable** (*i.e.*, all poles on left-hand side or at the origin) for these calculations to apply

System types and steady-state errors

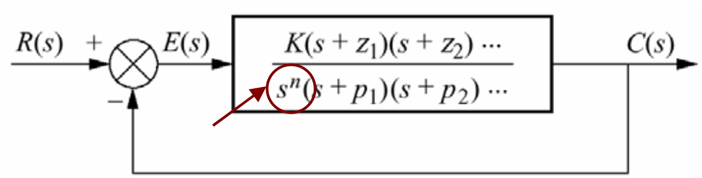
Table removed due to copyright restrictions.

Please see: Table 7.2 in Nise, Norman S. *Control Systems Engineering*. 4th ed. Hoboken, NJ: John Wiley, 2004.

$$\mathcal{L}\left[u(t)\right] = \frac{1}{s} \qquad K_p = \lim_{s\to 0} G(s)$$

$$\mathcal{L}\left[tu(t)\right] = \frac{1}{s^2} \qquad K_v = \lim_{s\to 0} sG(s)$$

$$\mathcal{L}\left[\frac{1}{2}t^2u(t)\right] = \frac{1}{s^3} \qquad K_a = \lim_{s\to 0} s^2G(s)$$

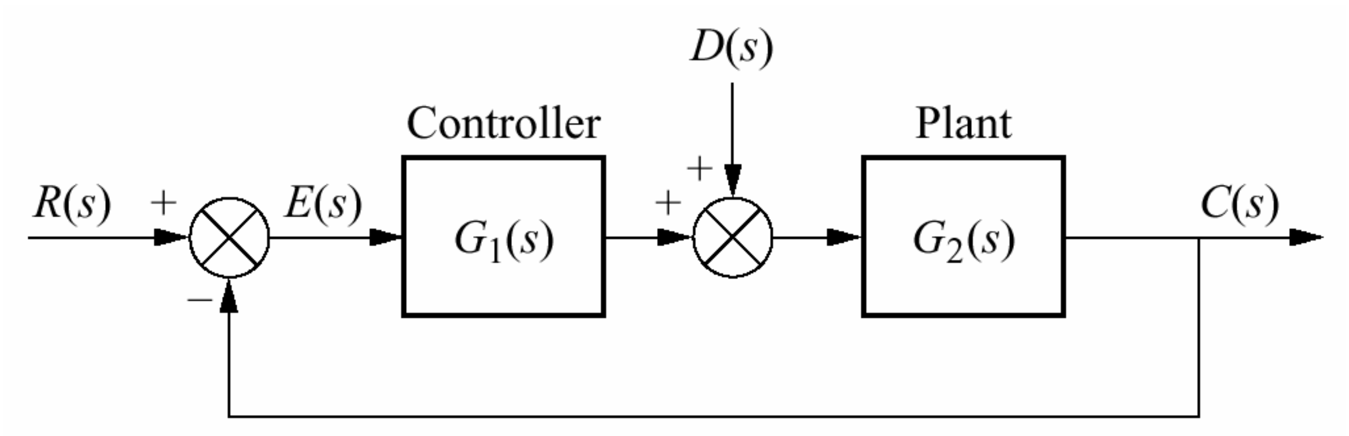


$n = 0$ Type 0

$n = 1$ Type 1

$n = 2$ Type 2

Disturbances



From the I–O relationship of the plant,

$$C(s) = E(s)G_1(s)G_2(s) + D(s)G_2(s).$$

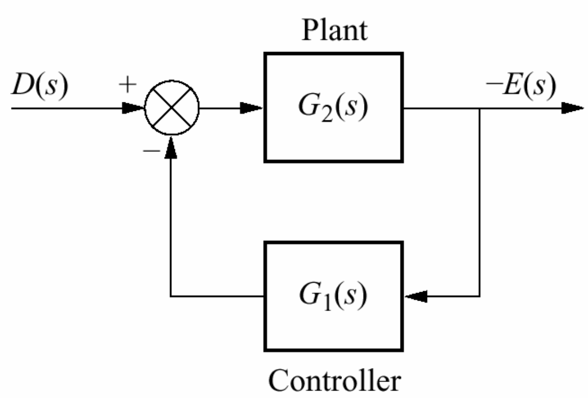
From the summation element,

$$E(s) = R(s) - C(s).$$

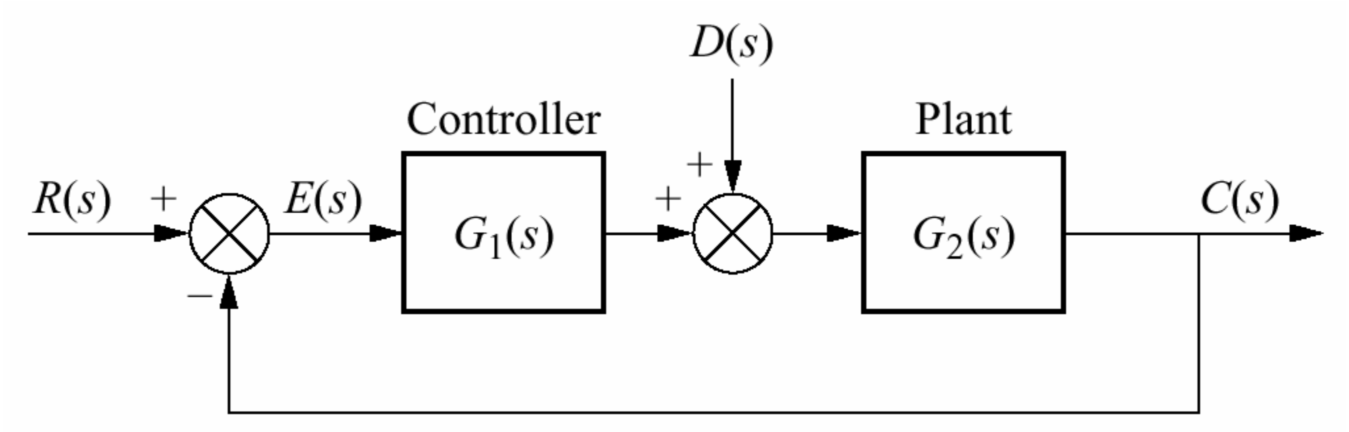
Substituting $C(s)$ and solving for $E(s)$,

$$E(s) = R(s)\frac{1}{1 + G_1(s)G_2(s)} - D(s)\frac{G_2(s)}{1 + G_1(s)G_2(s)}$$

Equivalent block diagram
with $D(s)$ as input
and $-E(s)$ as output.



Disturbances

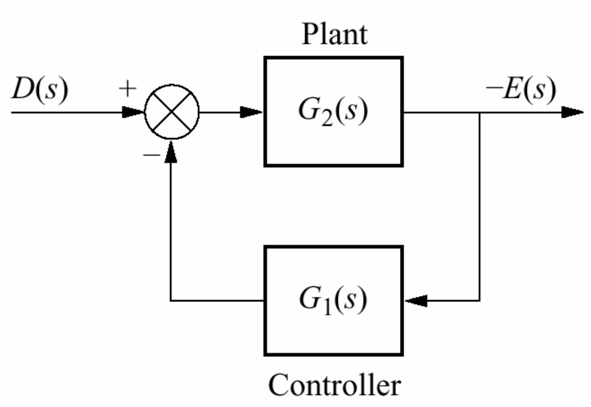


$$e(\infty) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\left[R(s)\frac{1}{1+G_1(s)G_2(s)} - D(s)\frac{G_2(s)}{1+G_1(s)G_2(s)}\right] \equiv e_R(\infty) + e_D(\infty),$$

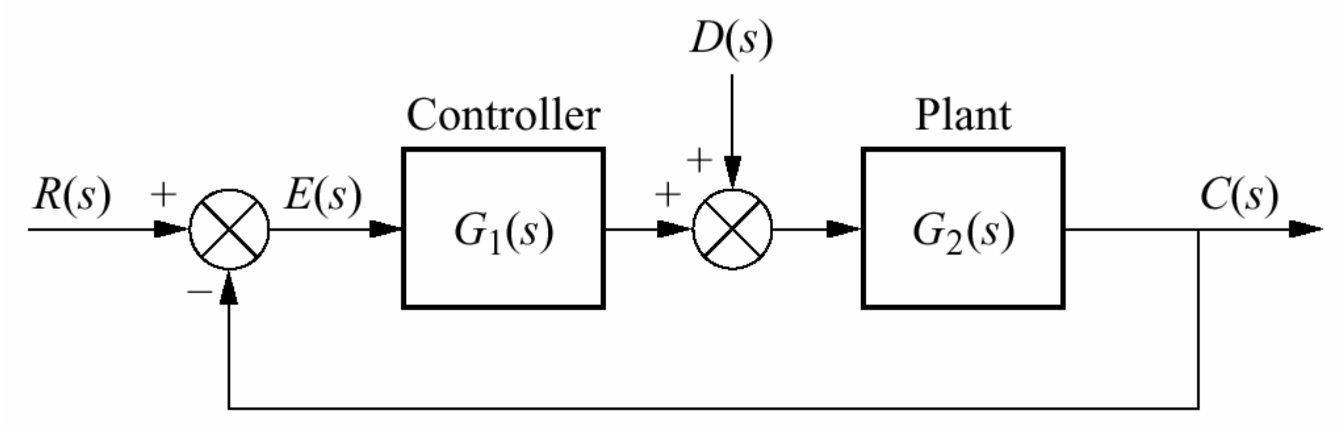
where

$$e_R(\infty) = \lim_{s\to 0}\frac{sR(s)}{1+G_1(s)G_2(s)}$$

$$e_D(\infty) = -\lim_{s\to 0}\frac{sG_2(s)D(s)}{1+G_1(s)G_2(s)}.$$

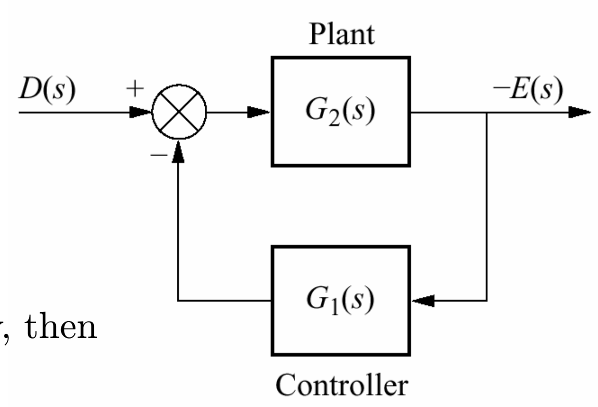


Unit step disturbance



Special case: unit step disturbance $D(s) = \frac{1}{s}.$

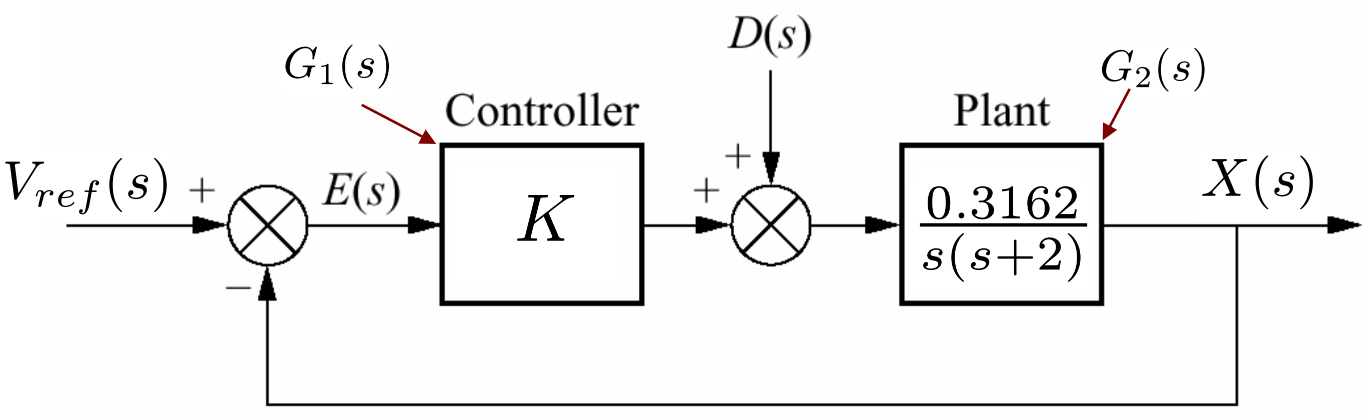
$$
\begin{aligned}
e_D(\infty) &= -\lim_{s\to 0} \frac{sG_2(s)\times(1/s)}{1+G_1(s)G_2(s)} \\
&= -\frac{\lim_{s\to 0} G_2(s)}{1+\lim_{s\to 0} G_1(s)G_2(s)} \\
&= -\frac{1}{\dfrac{1}{\lim_{s\to 0} G_2(s)} + \lim_{s\to 0} G_1(s)}.
\end{aligned}
$$



If K_1, K_2 are the gains of the controller and plant, respectively, then

$$e(\infty)\downarrow \quad \text{if} \quad K_1\uparrow \quad \text{or} \quad K_2\downarrow .$$

Unit step disturbance: example



DC motor with rack–pinion load, position feedback,

subject to unit step disturbance $D(s) = \frac{1}{s}$.

$$
\begin{aligned}
e_D(\infty) &= -\frac{1}{\dfrac{1}{\lim_{s\to 0} G_2(s)} + \lim_{s\to 0} G_1(s)} \\
&= -\frac{1}{\lim_{s\to 0} \dfrac{s(s+2)}{0.3162} + \lim_{s\to 0} K} \\
&= -\frac{1}{K}.
\end{aligned}
$$