MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Mechanical Engineering

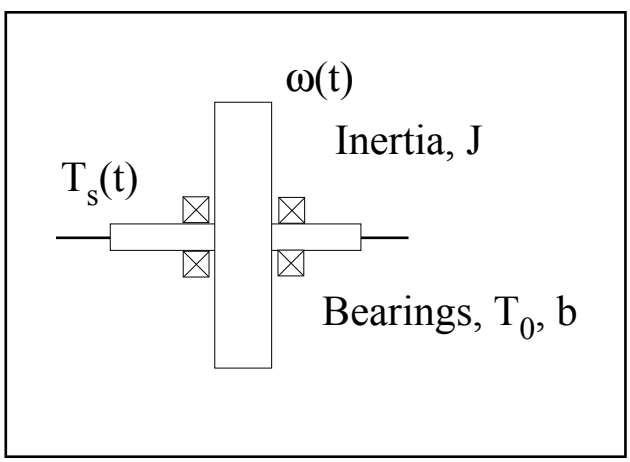
2.004 Dynamics and Control II
Fall 2007

Lecture 2
Solving the Equation of Motion

Goals for today

- Modeling of the 2.004 Lab's rotational system
- Analytical solution of the equation of motion for a 1st–order system using the time domain
- **Next lecture (Monday):** Solution of the equations of motion in the Laplace domain (s–domain).

As we saw in lecture 1, the Equation of Motion of a mechanical system is, in general, an Ordinary Differential Equation (ODE). In this lecture, we will remind ourselves how to solve ODEs analytically in the time domain and in MATLAB (*i.e.*, numerically.) We will consider the following rotational system (plant) of a motor attached to a shaft with viscous and/or Coulomb friction:



The motor applies torque $T_s(t)$ which is zero for $t < 0$ and increases to a step T_0 at $t = 0$. The shaft inertia is J while we assume that the motor inertia is negligible. We will also neglect the system compliance. *(Justify these assumptions.)* As input to the system we consider the torque, while the output is the angular velocity.

We will consider three cases for the friction applied by the bearings: *(i)* viscous friction of coefficient $f_v \equiv b$ (units $\mathrm{N \cdot m \cdot sec}$); *(ii)* Coulomb friction of magnitude f_c (units $\mathrm{N \cdot m \cdot sec}$); and *(iii)* both viscous and Coulomb friction.

1. Viscous friction

The system equation of motion is

$$J\frac{\mathrm{d}\omega}{\mathrm{d}t} + b\omega = T_\mathrm{s}(t) \qquad \Rightarrow \qquad \frac{J}{b}\dot{\omega} + \omega = \frac{1}{b}T_\mathrm{s}(t). \tag{1}$$

This is a <u>linear 1st–order ODE with constant coefficients</u>. We are interested in the evolution of the system's output (angular velocity) after application of the input (motor torque) at $t = 0$. In general, the solution is the sum of two terms, the <u>homogeneous</u> term and the <u>forced</u> term:

$$\omega(t) = \omega_\mathrm{h}(t) + \omega_\mathrm{f}(t) \tag{2}$$

The homogeneous term is due to the initial conditions in the system. In this case, a non–zero initial condition $\omega(t = 0)$ means that the wheel had been spinning before application of the torque by the motor at $t = 0$. The homogeneous term in linear ODEs decays exponentially to zero ("spin–down") if the system is stable. We will discuss more about stability in Lecture 14.

The homogeneous response of a linear ODE with constant coefficients consists of as many exponential terms as the order of the ODE. In this case, the ODE is 1st–order, so the homogeneous response is

$$\omega_\mathrm{h}(t) = A\mathrm{e}^{-t/\tau}. \tag{3}$$

To find the time constant τ we set the input torque $T_\mathrm{s} = 0$ and substitute (3) into the unforced equation of motion. Thus, we obtain

$$-\frac{J}{b}\frac{1}{\tau} + 1 = 0 \Rightarrow \tau = \frac{J}{b}.$$

The coefficient A is to be obtained later from the initial conditions.

The forced term is due to the excitation or input of the system, as the name implies. In this case, the input is the torque applied by the motor. It is expressed as a "step function:"

$$T_\mathrm{s}(t) = \begin{cases} 0, & t < 0 \\ T_0, & t \geq 0 \end{cases} \equiv T_0 u(t). \tag{4}$$

To find the forced response, we observe that in response to a step input the angular velocity should also approach a constant value after a sufficiently long time has elapsed (*i.e.*, as $t \to \infty$). This long–term constant value is called the "steady state" response of the system, and it also constitutes the forced response. It is found by setting the derivative term(s) in the equation of motion to equal zero. From (1) we find

$$\omega(\infty) = \omega_\mathrm{f}(t) = \frac{T_0}{b}. \tag{5}$$

Note the units for $\omega(\infty)$:

$$\frac{[\mathrm{N}\cdot\mathrm{m}]}{[\mathrm{N}\cdot\mathrm{m}\cdot\mathrm{sec}]} = \frac{1}{[\mathrm{sec}]} \equiv [\mathrm{Hz}]\,,$$

as it should be.

The complete response is, therefore,

$$\omega(t) = \omega_{\mathrm{f}}(t) + \omega_{\mathrm{h}}(t) = \frac{T_0}{b} + A\mathrm{e}^{-t/\tau}$$

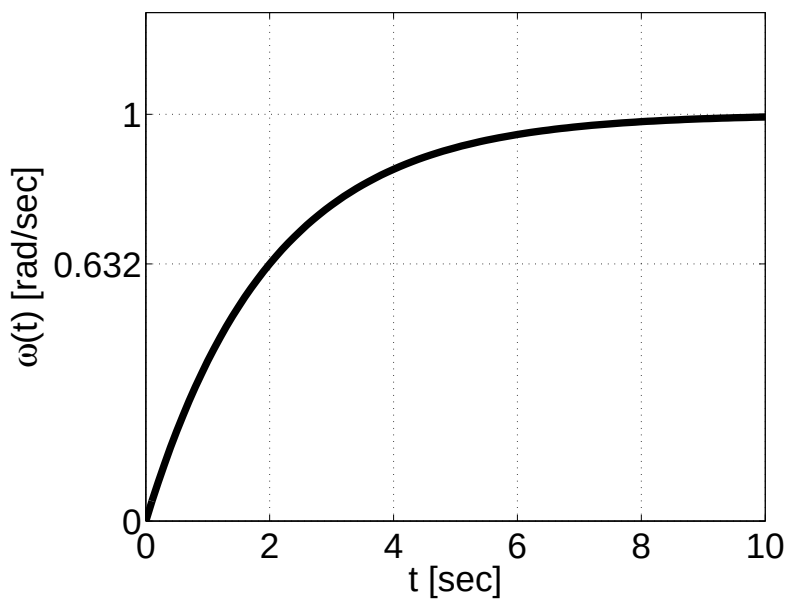
All that's left is to find the unknown coefficient A. Since the coefficient appears in the homogeneous part of the response, it must be determined by the initial conditions. For simplicity, we assume that the shaft was initially at rest: $\omega(t = 0) = 0$. Substituting, we find

$$\frac{T_0}{b} + A = 0 \Rightarrow A = -\frac{T_0}{b},$$

and finally

$$\omega(t) = \frac{T_0}{b}\left(1 - \mathrm{e}^{-t/\tau}\right). \tag{6}$$

The result is plotted below for numerical values $J = 4.0\,\mathrm{kg{\cdot}m}$, $b = 2.0\,\mathrm{N{\cdot}m\cdot sec}$, and $T_0 = 2.0\,\mathrm{N{\cdot}m}$. We can see that the time constant equals the time it takes for the system to reach angular velocity equal to $1 - \mathrm{e}^{-1} = 63.2\%$ of its steady–state value $\omega(\infty)$. Within 5 time constants, the system reaches $1 - \mathrm{e}^{-5} = 99.3\%$ of its steady–state value.



2. Coulomb and viscous friction

The system equation of motion is

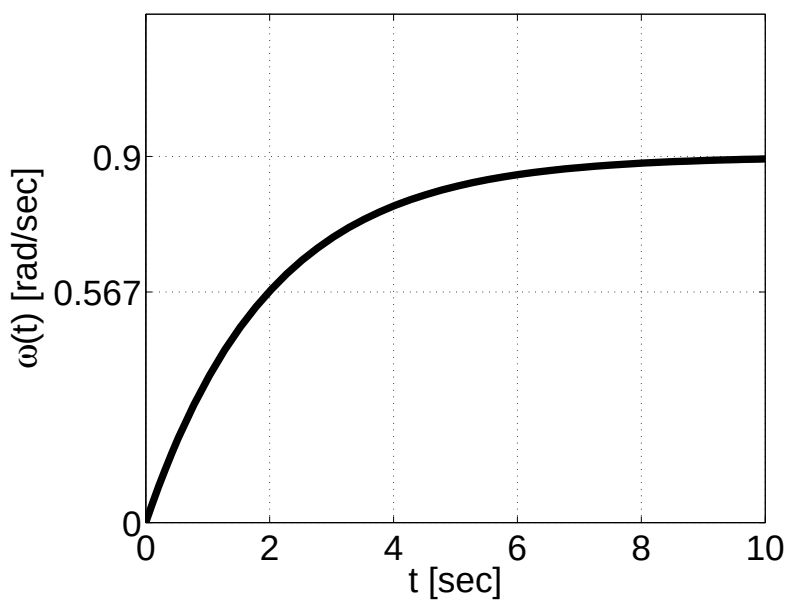
$$J\frac{\mathrm{d}\omega}{\mathrm{d}t} + b\omega + f_c \operatorname{sgn}(\omega) = T_\mathrm{s}(t) \qquad \Rightarrow \qquad \tau\dot{\omega} + \omega = \frac{T_\mathrm{s} - f_c \operatorname{sgn}(\omega)}{b}, \tag{7}$$

where τ is given by (3) as before. We note that (1) and (7) are almost identical with one difference: assuming $\omega > 0$, Coulomb friction effectively reduces the torque applied by the motor. We can use the method described in the previous section to solve (7) and obtain

$$\omega(t) = \frac{T_0 - f_c}{b}\left(1 - \mathrm{e}^{-t/\tau}\right). \tag{8}$$

Note that our solution is valid because $\omega(t) > 0$ for all $t > 0$.

The result is plotted below for the same numerical values as in section 1, and $f_c = 0.2\ \mathrm{N}\cdot\mathrm{m}$. We can see that the steady–state value is also reduced by Coulomb friction but the time constant remains the same as in the case of viscous friction only.



3. Numerical solution of the equations of motion

When the equations of motion cannot be solved neatly analytically as in the two previous examples, we must resort to numerical solution. In MATLAB, the ODE solver is the built–in function `ode45` which works in systems that we will consider in this class. At present, we will describe how to use `ode45` to solve 1st–order ODEs. We will see later how to solve higher order ODEs.

For 1st–order ODEs, one must first solve the ODE for the derivative; e.g.

$$\dot{\omega} = \frac{1}{J}\Big(T_s(t) - b\omega - f_c \,\mathrm{sgn}\,(\omega)\Big)$$

in the example of Coulomb and viscous friction (7). Then one must define a function that returns the value of $\dot{\omega}$ with variables t and ω and use this function as argument when calling `ode45`. In the class website, you will find two functions that show how this is done:

- `shaftcv_kernel` returns $\dot{\omega}$ as function of t, ω and the parameters J, b, f_c, T_0;
- `shaftcv_solve` calls `ode45` with the additional information of the simulation timespan [0 `tmax`] and the initial condition $\omega(t = 0) =$ `omega_0`. Note that we call `ode45` in a slightly sophisticated way that allows us to pass J, b, f_c, T_0 as parameters; MATLAB's standard way would have required these parameters to be assigned fixed numerical values inside `shaftcv_kernel`.

Script `shaft_w_coulomb_viscous` shows how to call `shaftcv_solve` with specific numerical values for the various parameters and plot the results. The numerical solution agrees with the plot shown in the previous page.

You can easily modify the three functions to solve more complicated ODEs numerically, e.g. with more complicated nonlinear terms, a torque $T_\mathrm{s}(t)$ that is not a step function, etc.