Solutions for Problem Set 5

Problem 1. Particle slides down movable inclined plane. The inclined plane of mass M is constrained to move parallel to the X-axis, and the particle of mass m is constrained to remain on the sloping surface of the inclined plane.

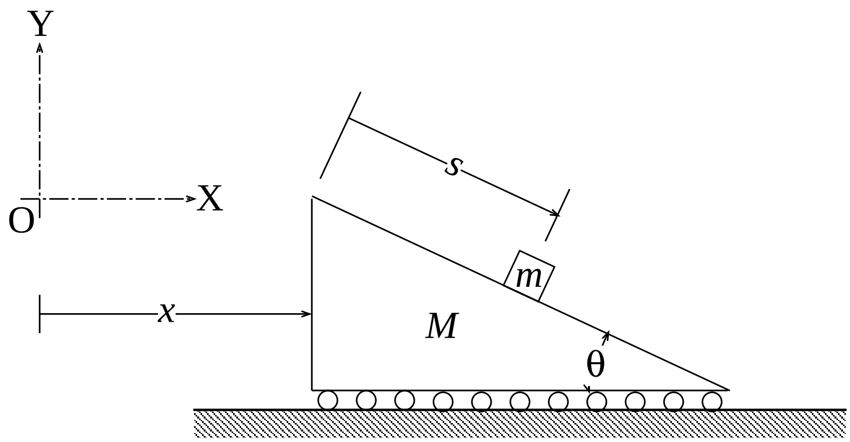

Figure 1: Mass m slides down inclined plane of mass M.

(a) The imclined plane M is located by the coordinate x, and once the position of the inclined plane is fixed the location of the mass partiicle m is determined by giving the distance s down the slope. The coordinates x and s constitute a set of complete and independent generalized coordinates for the system under consideration.

(b) To derive the equations of motion for the generalized coordinates x and s we begin by studying the motion. The particle m translates in both the X- and Y- directions while the inclined plane M translates in just the X-direction. There is no rotation. The X- and Y-coordinates of the particle m are

$$x_m = x + s\cos\theta \qquad \text{and} \qquad y_m = -s\sin\theta$$

so its velocity coordinates are

$$\dot{x}_m = \dot{x} + \dot{s}\cos\theta \qquad \text{and} \qquad \dot{y}_m = -\dot{s}\sin\theta$$

The velocity of the inclined plane M is $\dot{x}_M = \dot{x}$.

Next, to study the forces, we draw separate free-body diagrams of the paricle m and the inclined plane M. Since there is no friction, the reaction forces N_1 and N_2 are *normal* to the surfaces making contact. Note that the force N_1 acting on the inclined plane M is equal and opposite to the force N_1 acting on the particle m.

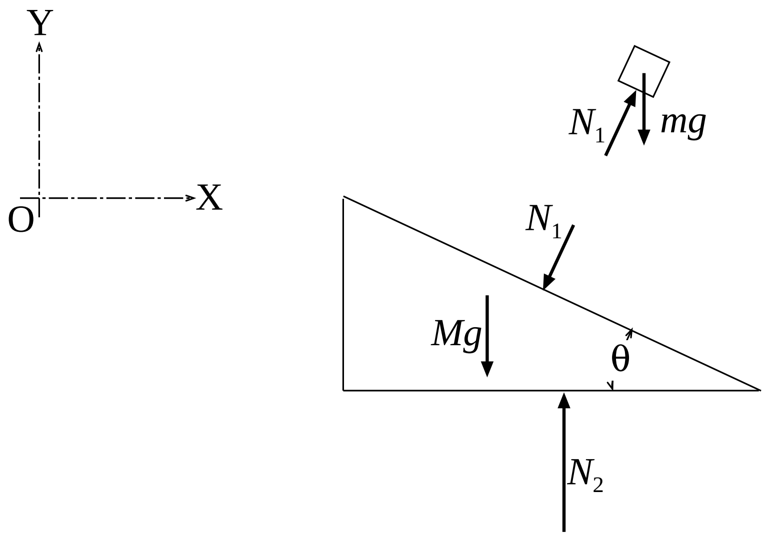

Figure 2: Forces acting on particle m and inclined plane M.

(b) The equations of motion are obtained by applying the momentum principles to the particle m and the inclined plane M. For either a particle or a single rigid body, the linear momentum principle requires that the vector sum of the forces acting on the object equals the time rate of change of the objects's linear momentum vector. In terms of the x- and y-components of the vectors involved

$$\sum f_x = \frac{dp_x}{dt} \qquad \text{and} \qquad \sum f_y = \frac{dp_y}{dt}$$

For the particle m, the x- and y-components of linear momentum are

$$p_x = m\dot{x}_m = m(\dot{x} + \dot{s}\cos\theta) \qquad \text{and} \qquad p_y = m\dot{y}_m = -m(\dot{s}\sin\theta)$$

and the force components acting on it are

$$\sum f_x = N_1 \sin\theta \qquad \text{and} \qquad \sum f_y = N_1\cos\theta - mg$$

so that the results of applying the linear momentum principle to the particle are the two equations

$$N_1 \sin\theta = m(\ddot{x} + \ddot{s}\cos\theta) \tag{1}$$

$$N_1\cos\theta - mg = -m\ddot{s}\sin\theta \tag{2}$$

For the inclined plane M, the constraints do not allow vertical motion, so only the horizontal components of the momentum principle need to be considered. The horizontal linear momentum is $P_x = M\dot{x}$ and the resultant horizontal force on the inclined plane is $-N_1\sin\theta$. The momentum principle requires

$$-N_1\sin\theta = M\ddot{x} \tag{3}$$

Two equations of motion for the generalized coordinates x and s are obtained by eliminating the reaction force N_1 from these three equations to get two independent equations. The equation

$$(M+m)\ddot{x} + m\cos\theta\ddot{s} = 0$$

is obtained by simply adding Eq.(3) to Eq.(1). An independeent equation is obtained by multiplying Eq.(1) by $\cos\theta$, and multiplying Eq.(2) by $-\sin\theta$, and then adding these two to get

$$m\ddot{x}\cos\theta + m\ddot{s} = mg\sin\theta$$

The last two equations can be written neatly as a matrix equation

$$\begin{bmatrix} \dfrac{M}{m}+1 & \cos\theta \\ \cos\theta & 1 \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{s} \end{Bmatrix} = \begin{Bmatrix} 0 \\ g\sin\theta \end{Bmatrix}$$

Problem 2. Disk rolls on cylindrical surface. The sketch in Fig.3 shows a disk of radius r and mass m, which started from the position indicated by the dashed circle and then rolled through the angle θ to arrive at the position indicated by the solid circle. In the original position the center of the disk was at C on the vertical axis OY. After rolling through the angle θ on the cylindrical surface, the center of the disk is at C$'$.

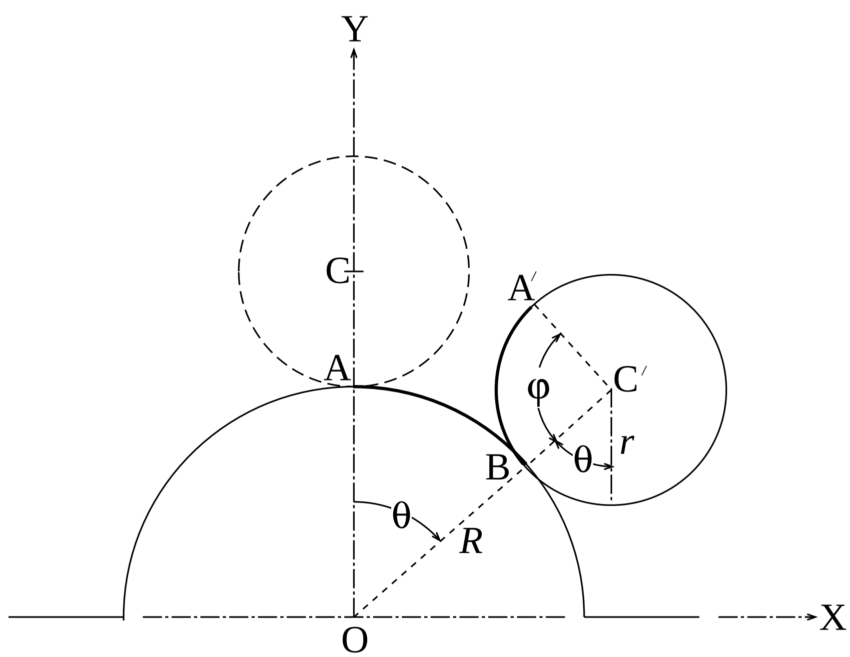

Figure 3: Disk of radius r rolls on fixed cylindrical surface of radius R.

Because there is no slip, the length of the arc A$'$B $= r\varphi$ on the disk, must be the same as the length of the path AB $= R\theta$. This implies that

$$\varphi = \frac{R}{r}\theta$$

(a) The angular velocity ω of the disk is the rate of turning of a stripe painted on the disk with respect to a fixed reference direction. Consider the stripe CA on the disk in its original position, indicated by the dashed circle. Afer the disk has rolled to its position indicated by the solid circle, the stripe is now in the position C$'$A$'$. Originally the stripe was vertical, but now the stripe C′A′ makes the angle $\varphi+\theta$ with the vertical. The angular velocity of the disk is

$$\omega = \frac{d}{dt}(\varphi + \theta) = \frac{R+r}{r}\dot{\theta}$$

(b) The kinetic energy of a rigid body can be obtained by evaluating the formula

$$KE = \frac{1}{2}mv_C^2 + \frac{1}{2}I_C\omega^2 \tag{1}$$

where v_C is the magnitude of the velocity of the mass center C and I_C is the moment of inertia of the rigid body about its mass center. For a uniform solid disk of radius r and mass m, $I_C = \frac{1}{2}mr^2$. The velocity of the mass center C can be obtained by applying the general formula

$$\vec{v}_C = \vec{v}_B + \vec{\omega}\times \overrightarrow{BC}$$

which applies to any two points B and C on the same rigid body which rotates with angular velocity $\vec{\omega}$. In the present case the point B on the disk is instantaneously at rest, so $\vec{v}_B = 0$, and the length of the vector $\overrightarrow{BC}$ is r, so the vector $\vec{v}_C$ has the magnitude $r\omega$ and is directed at right angles to BC. Substitution of $v_C = r\omega$ and $I_C = \frac{1}{2}mr^2$ into Eq.(1) yields

$$KE = \frac{1}{2}m(r\omega)^2 + \frac{1}{2}(\frac{1}{2}mr^2)\omega^2 = \frac{3}{4}m(r\omega)^2$$

Problem 3. Rod falls under the influence of gravity. In Fig.4 the initial position of the rod is shown along with an inertial reference frame XOY with its origin O placed at the initial contact point of the end B of the rod.

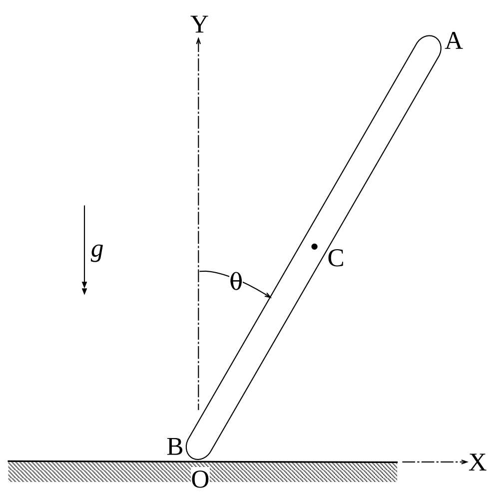

Figure 4: Rod of mass m and length L slides on floor as it falls.

The rod is completely located by the giving the coordinates $x_C$and y_C of the mass center C and the angle θ that the rod makes with the vertical. However these coordinates are not independent because the constraint that the end B always remains in contact with the floor requires that the relation

$$y_c = \frac{L}{2}\cos\theta$$

always be satisfied.

(a) The constrained system has only two degrees of freedom. One independent set of coordinates is x_C and θ, with the dependent variable $y_c = \frac{L}{2}\cos\theta$. Another set of independent coordinates is x_C and y_C, with the dependent variable $\theta = \cos^{-1}\frac{2y_C}{L}$. The subsequent algebra is somewhat simpler with the first choice.

(b) We study the motion, using the generalized coordinates x and θ. From the displacement components of the mass center C,

$$x_C = x \qquad \text{and} \qquad y_C = \frac{L}{2}\cos\theta,$$

the velocity components are obtained by differentiation

$$\dot{x}_C = \dot{x} \qquad \text{and} \qquad \dot{y}_C = -\frac{L}{2}\dot{\theta}\sin\theta$$

The angular velocity of the rod is $\omega = \dot{\theta}$ in the clockwise direction.

Next, we study the forces by drawing a free-body diagram of the rod showing all the forces acting on it. See Fig.5.

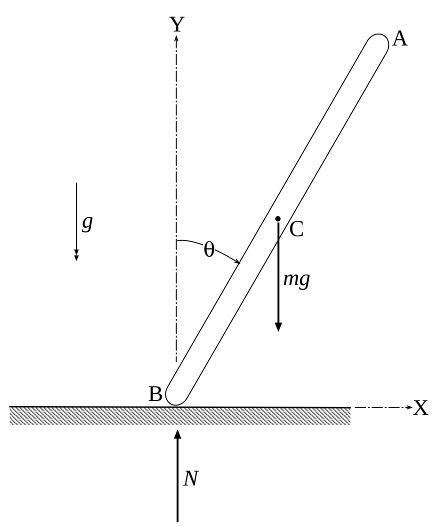

Figure 5: Rod is acted on by gravity force mg and floor reaction N.

There are no horizontal forces acting on the rod, so the horizontal momentum is conserved. Since the rod is at rest when it is released at $t = 0$ this means that the mass center C does not move horizontally while the rod falls. The equation of motion for the generalized coordinate x is

$$x = \frac{L}{2}\sin\frac{\pi}{6} = \frac{L}{4}, \text{ a constant}$$

Application of the linear momentum principle in the vertical direction produces the equation

$$\sum f_y = N - mg = \frac{d}{dt}(m\dot{y}_C) = -m\frac{L}{2}(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta) \quad \text{or} \quad N = m[g - \frac{L}{2}(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta)] \tag{1}$$

and application of the angular momentum principle about the mass center C produces the equation

$$\tau = N\frac{L}{2}\sin\theta = \frac{dH_C}{dt} = \frac{d}{dt}(m\frac{L^2}{12}\omega) = m\frac{L^2}{12}\ddot{\theta} \tag{2}$$

Elimination of the reaction force N between (1) and (2) yields the equation of motion for the generalized coordinate θ

$$m\frac{L^2}{2}\sin\theta[\frac{g}{L} - \frac{1}{2}(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta)] = m\frac{L^2}{12}\ddot{\theta} \quad \text{or} \quad \ddot{\theta}(1 + 3\sin^2\theta) + 3\dot{\theta}^2\sin\theta\cos\theta = 6\frac{g}{L}\sin\theta \tag{3}$$

(c) Immediately after the rod is released from rest the variable θ still has its initial value $\theta(0) = \pi/6$ and the angular velocity $\omega = \dot{\theta}$ still has its initial value $\omega(0) = 0$. Inserting these initial values into (3) yields

$$\ddot{\theta}(0)[1 + \frac{3}{4}] = 3\frac{g}{L} \qquad \text{or} \qquad \ddot{\theta}(0) = \frac{12}{7}\frac{g}{L} \tag{4}$$

(d) The initial value of the floor reaction force N is obtained by substituting the initial values $\theta(0) = \pi/6$, $\dot{\theta}(0) = 0$ and $\ddot{\theta}(0) = \frac{12}{7}\frac{g}{L}$ into (1) to get

$$N(0) = mg(1 - \frac{1}{2}\frac{12}{7}\frac{1}{2}) = \frac{4}{7}mg$$