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2.004 Dynamics and Control II
Spring 2008

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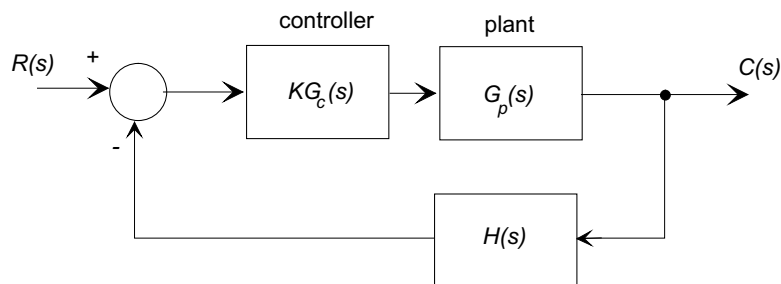
Lecture 27¹

Reading:

- Nise: Chapter 8

1 Root Locus Development

In Lecture 26 we saw some simple examples of root locus plots. We now look at the general method of generating root loci.



Recall that the closed-loop characteristic equation is:

$$1 + KG(s) = 0$$

where $G(s) = G_c(s)G_p(s)H(s)$ is the open-loop transfer function. We now ask ourselves: how can we tell if an arbitrary point $s = \sigma + j\omega$ lies on the root locus? In other words we seek conditions that determine whether s is a root of the characteristic equation. From above, s is a root if

$$KG(s) = -1 + j0.$$

In polar form this may be expressed as

$$\begin{aligned} KG(s) &= |KG(s)| e^{j\angle(KG(s))} \\ &= 1 \times e^{j(2n+1)\pi} \quad \text{for } n = 0, 1, 2, \dots \\ &= \cos((2n + 1)\pi) + j \sin((2n + 1)\pi) \\ &= -1 + j0. \end{aligned}$$

This tells us that for any point $s = \sigma + j\omega$ on the root locus

$$|KG(s)| = 1 \quad \text{and} \quad \angle(G(s)) = (2n + 1)\pi$$

which generates two important conditions:

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The Angle Condition: $\angle(G(s)) = (2n + 1)\pi$

The Magnitude Condition: $|KG(s)| = 1$

In practice, the angle condition is used to determine whether a point s lies on the root locus, and if it does, the magnitude condition is used to determine the gain K associated with that point, since $K = 1/|G(s)|$.

■ Example 1

Given the open-loop system

$$G(s) = \frac{1}{(s + 2)(s + 4)}$$

determine whether the points $s = -1$, $s = -3.5$, $s = -3 + j5$ are on the root locus.

For $s = -1$:

$$KG(s) \frac{K}{(-1 + 2)(-1 + 4)} = \frac{K}{3}$$

$KG(s) \neq -1$ for any $K > 0$, so we conclude $s = -1$ is not on the root-locus.

For $s = -3.5$:

$$KG(s) \frac{K}{(-3.5 + 2)(-3.5 + 4)} = \frac{K}{-0.75}$$

$KG(s) = -1$ for $K = 0.75$, so we conclude $s = -3.5$ lies on the root-locus.

For $s = -3 + j5$:

$$KG(s) = \frac{K}{((-3 + j5) + 2)((-3 + j5) + 4)} = \frac{K}{(-1 + j5)(1 + j5)} = \frac{K}{-26}$$

$KG(s) = -1$ for $K = 26$, so we conclude $s = -3 + j5$ lies on the root-locus.

1.1 Geometric Evaluation of the Transfer Function

The transfer function may be evaluated for any value of $s = \sigma + j\omega$, and in general, when s is complex the function $G(s)$ itself is complex. It is common to express the complex value of the transfer function in polar form as a magnitude and an angle:

$$G(s) = |G(s)| e^{j\angle G(s)},$$

with a magnitude $|G(s)|$ and an angle $\angle G(s)$ given by

$$\begin{aligned} |G(s)| &= \sqrt{\Re\{G(s)\}^2 + \Im\{G(s)\}^2}, \\ \angle G(s) &= \tan^{-1} \left(\frac{\Im\{G(s)\}}{\Re\{G(s)\}} \right) \end{aligned}$$

where $\Re \{ \}$ is the real operator, and $\Im \{ \}$ is the imaginary operator.

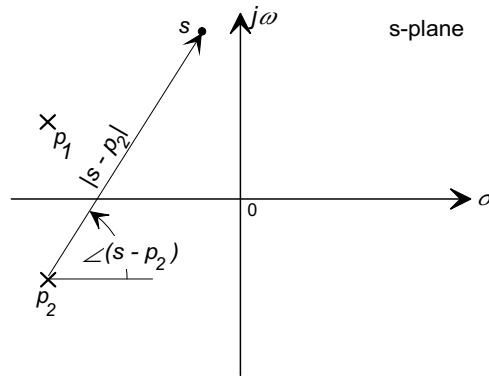
If the numerator and denominator polynomials are factored into terms $(s-p_i)$ and $(s-z_i)$,

$$G(s) = C \frac{(s-z_1)(s-z_2)\dots(s-z_{m-1})(s-z_m)}{(s-p_1)(s-p_2)\dots(s-p_{n-1})(s-p_n)},$$

(where C is a constant), each of the factors in the numerator and denominator is a complex quantity, and may be interpreted as a *vector* in the s -plane, originating from the point z_i or p_i and directed to the point s at which the function is to be evaluated. Each of these vectors may be written in polar form in terms of a magnitude and an angle, for example for a pole $p_i = \sigma_i + j\omega_i$, the magnitude and angle of the vector to the point $s = \sigma + j\omega$ are

$$\begin{aligned} |s-p_i| &= \sqrt{(\sigma-\sigma_i)^2 + (\omega-\omega_i)^2}, \\ \angle(s-p_i) &= \tan^{-1} \left(\frac{\omega-\omega_i}{\sigma-\sigma_i} \right) \end{aligned}$$

as shown below.



Because the magnitude of the product of two complex quantities is the product of the individual magnitudes, and the angle of the product is the sum of the component angles, that is if a and b are complex, that is

$$\begin{aligned} |ab| &= |a| |b|, & |a/b| &= |a| / |b| \\ \angle(ab) &= \angle a + \angle b, & \angle(a/b) &= \angle a - \angle b \end{aligned}$$

the magnitude and angle of the complete transfer function may then be written

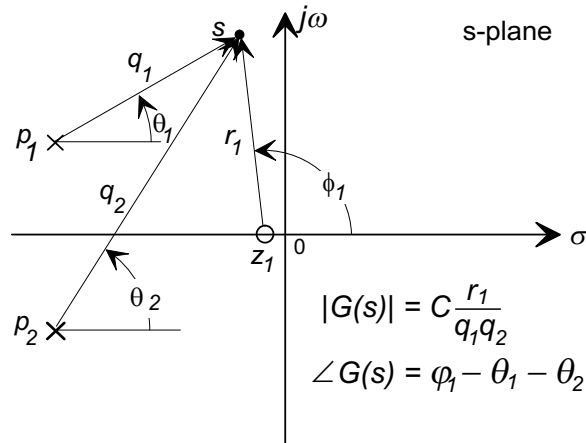
$$\begin{aligned} |G(s)| &= C \frac{\prod_{i=1}^m |(s-z_i)|}{\prod_{i=1}^n |(s-p_i)|} \\ \angle G(s) &= \sum_{i=1}^m \angle(s-z_i) - \sum_{i=1}^n \angle(s-p_i). \end{aligned}$$

The magnitude of each of the component vectors in the numerator and denominator is the distance of the point s from the pole or zero on the s -plane. Therefore if the vector from the pole p_i to the point s on a pole-zero plot has a length q_i and an angle θ_i from the horizontal,

and the vector from the zero z_i to the point s has a length r_i and an angle ϕ_i , as shown above, the value of the transfer function at the point s is

$$|G(s)| = C \frac{r_1 \cdots r_m}{q_1 \cdots q_n}$$

$$\angle G(s) = (\phi_1 + \cdots + \phi_m) - (\theta_1 + \cdots + \theta_n)$$



The transfer function at any value of s may therefore be determined geometrically from the pole-zero plot, except for the overall “gain” factor C . The magnitude of the transfer function is proportional to the product of the geometric distances on the s -plane from each zero to the point s divided by the product of the distances from each pole to the point. The angle of the transfer function is the sum of the angles of the vectors associated with the zeros minus the sum of the angles of the vectors associated with the poles.

The angle condition then states that for a point $s = \sigma + j\omega$ to be on the root locus,

$$\angle G(s) = \sum_{i=1}^m \phi_i - \sum_{i=1}^n \theta_i = (2n + 1)\pi,$$

and once it has been established that s lies on the locus, the magnitude condition may be used to determine the value of K :

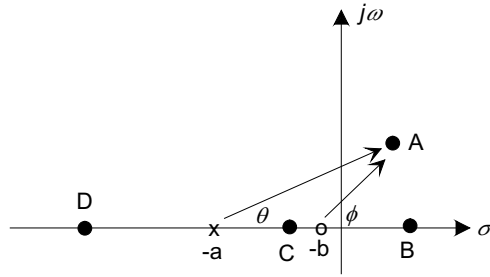
$$K = \frac{1}{|G(s)|}$$

■ Example 2

For open-loop system

$$G(s) = \frac{s + b}{s + a}$$

use the angle condition to determine whether the points labeled A , B , C and D lie on the root locus.



For an arbitrary point s

$$\angle G(s) = \phi - \theta.$$

At Point A: $\phi - \theta \neq (2n + 1)\pi$, therefore A is not on the root locus.

At Point B: $\phi = 0, \theta = 0, \phi - \theta = 0$, therefore B is not on the root locus.

At Point C: $\phi = \pi, \theta = 0, \phi - \theta = \pi$, therefore C is on the root locus.

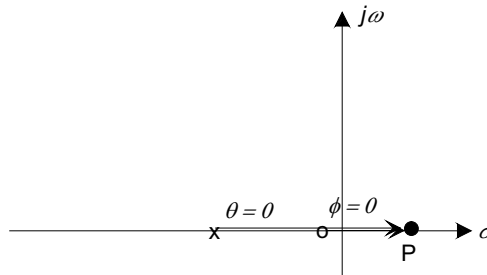
At Point D: $\phi = \pi, \theta = \phi, \phi - \theta = 0$, therefore D is not on the root locus.

The only one of these four points that satisfies the angle condition, and therefore lies on the root locus, is point C .

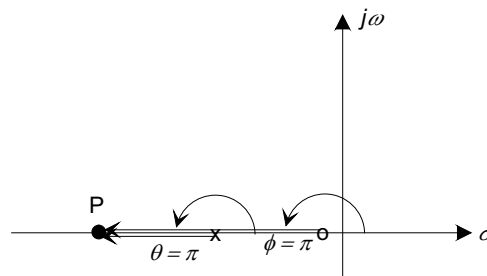
1.2 Regions of the Real Axis on the Root Locus

We note that for a point $s = \sigma$ that lies on the real axis:

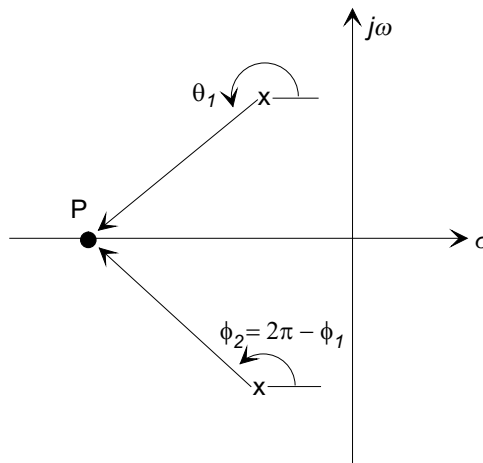
- (a) Poles and zeros on the real axis that lie to the *left of the point s* contribute zero to the angle condition.



- (b) Poles and zeros on the real axis that lie to the *right of the point s* contribute $\pm\pi$ to the angle condition.



- (c) A complex conjugate pole or zero pair contributes a total of zero (2π) to the angle condition along the real axis.



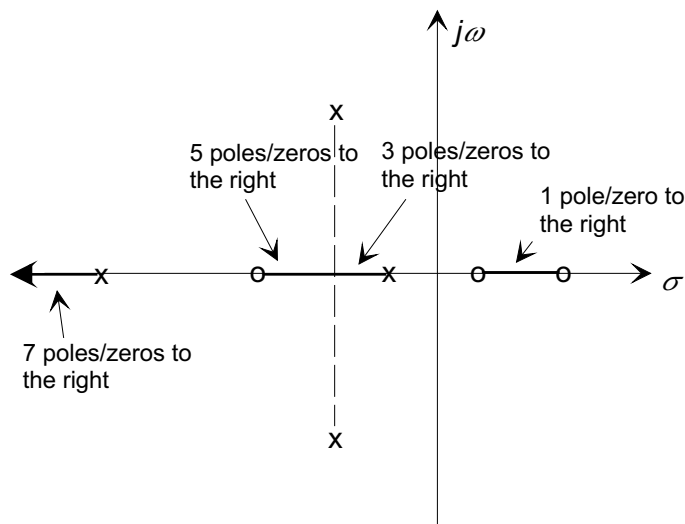
These observations combine to generate the condition:

A point on the real axis lies on the root locus only if there are an odd number of poles and/or zeros to its right.

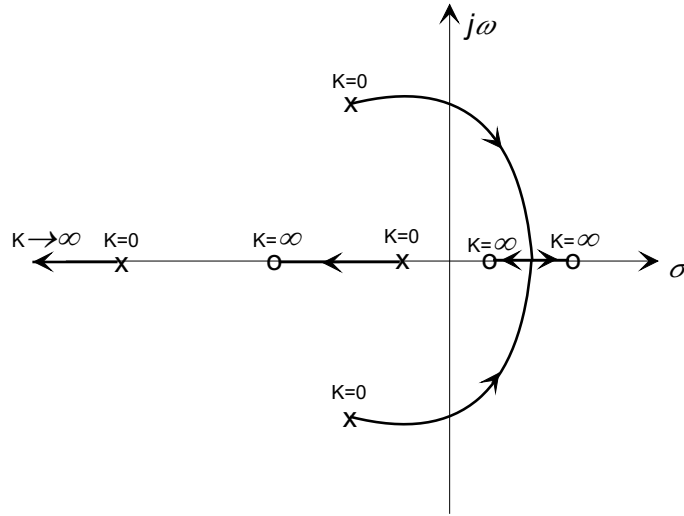
■ Example 3

Define the regions of the real axis that will lie on the root locus for the following open-loop pole-zero plot with 4 poles and 3 zeros, and then qualitatively fill in the rest of the plot.

The real axis regions are shown below:



Since we know that (1) branches must originate from open-loop poles, and (2) terminate on open-loop zeros or go to infinity, we have enough information to sketch the *form* of the root locus:



where we note that branches must originate from the complex conjugate pole pair and terminate on the r.h. plane zeros.

1.3 Behavior of the Root Locus for Large Values of K

Let the closed-loop transfer function be

$$G_{cl}(s) = \frac{KG(s)}{1 + KG(s)} = \frac{KN(s)}{D(s) + KN(s)}$$

where $N(s)$ is of order m , and $D(s)$ is of order n . We can see from the above expression that provided $n \geq m$ the closed-loop system will also be of order n . When K becomes large we can approximate the closed-loop characteristic equation as

$$KN(s) = 0$$

(which is of order m) and so we can state:

As the value of $K \rightarrow \infty$, m of the closed-loop poles approach the m open loop zeros.

This leaves $n - m$ closed-loop poles unaccounted for. Let's assume that for large K , where m of the poles are very close to the zeros, pole-zero cancellation has taken place and the characteristic equation becomes

$$1 + KG(s) \approx 1 + \frac{K}{\prod_i (s - p_i)} \approx 1 + \frac{K}{s^{n-m}} = 0$$

where the p_i are the $n - m$ uncanceled poles. With this approximation

$$s^{n-m} = -K, \quad \text{or} \quad s = K^{1/(n-m)}(-1)^{1/(n-m)}$$

The $n - m$ roots of -1 are complex with values

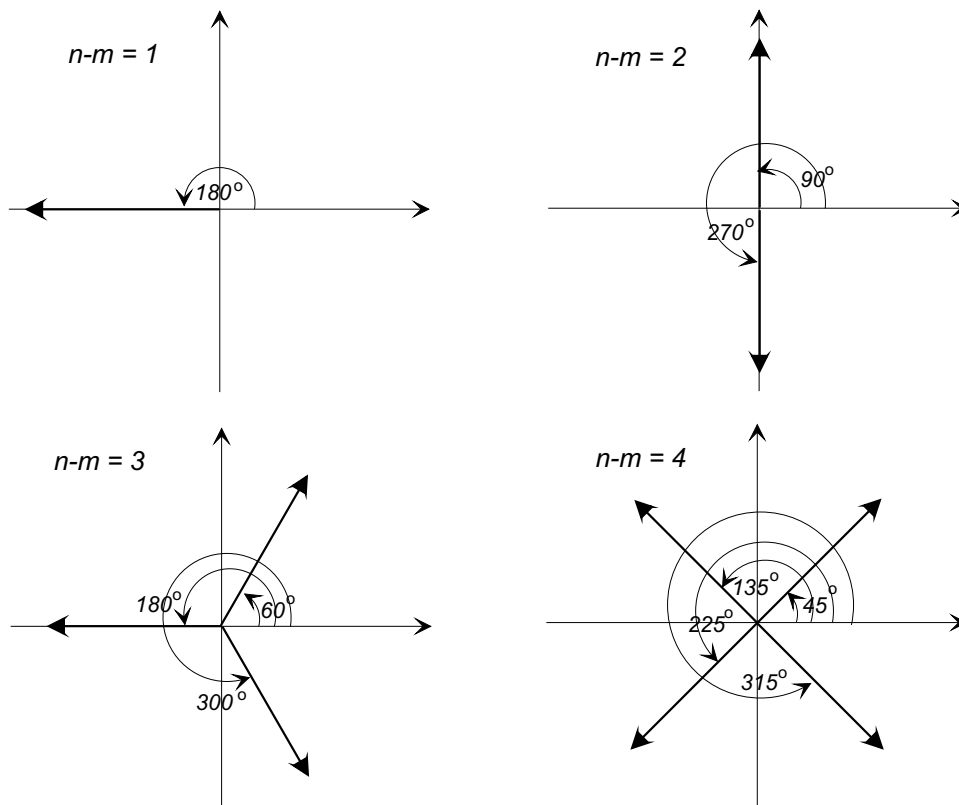
$$s_k = e^{j(2k+1)\pi/(n-m)} \quad k = 0, 1, \dots, n - m - 1$$

that is, they lie equally spaced around the unit circle at angles

$$\theta_k = \frac{(2k + 1)\pi}{n - m}, \quad k = 0, 1, \dots, n - m - 1.$$

and as K becomes large, the $n - m$ closed-loop poles approach a set of radial asymptotes at these angles. The asymptotic angles are summarized in the following table:

| $n - m$ | Asymptote Angles |
|---------|---|
| 1 | 180° |
| 2 | $90^\circ, 270^\circ$ |
| 3 | $60^\circ, 180^\circ, 300^\circ$ |
| 4 | $45^\circ, 135^\circ, 225^\circ, 315^\circ$ |



As the gain K becomes large, $n - m$ branches of the root locus diverge away from the origin and approach $n - m$ radial asymptotes, at angles $\theta_k = (2k + 1)\pi/(n - m)$, for $k = 0 \dots (n - m - 1)$.

This is not quite the full picture. We will investigate this further in the next lecture.