PROFESSOR: So Professor Gossard gave the lecture last week. I'm going to pick up where he left off. But let's talk about the concept questions from the homework you've been working on.

So the first one is our cart. You'd expect to be able eliminate the terms involving gravity in the equations of motion by choosing coordinates with respect to the static equilibrium position. So we've talked about that. And with this one does the restoring force on the pendulum, what makes it come back to zero after it's a damped out and hanging straight down.

AUDIENCE: Gravity.

PROFESSOR: What? Gravity does. And that gravity term varies with the torque that the gravity puts around the pivot is MgL over 2 sine theta. So the gravity term is involved with the motion variable theta.

So in this case, gravity is going to be involved in the natural frequency and in the equations of motion, no matter what, So you will not be able to eliminate the gravity terms.

Next, this one, this is a vibration isolation question. Will the addition of damping increase or reduce the vibration of the table in response to the floor motion at 30 Hertz? I guess this depends on what the natural frequency of the system is. But we're trying to do vibration isolation.

And presuming, if you read the problem, you're supposed to find a a stiffness such that you can reduce the response of the table by 12 dB, I think it said, from the motion of the floor. So that's something substantially less than 1. And you will be--

this one takes a, best described with a picture.

The transfer function for response of the table over the motion of the floor, the magnitude of that transfer function, that's just the ratio of x to y. That looks like this, if the damping is zero.

And as you add damping, all points cross right here. Some damping does that. More damping because of this. And in order to accomplish what's been described-- this is 1.0 here.

If you're trying to make this table respond less than the floor, you must be somewhere out here where you're below 1. So this is omega over omega n. And right here at resonance, you're at 1.0. So this is, you know, two or three or four for this value out here. Let's say here's where you find the answer to B.

And without damping, you're there. And that's 12 dB down. If you add damping, it pushes you up these curves. Does make the response larger, the undesirable response, the motion of the table larger or smaller as you add damping at that operating point? It increases it, right?

OK, so in this case, will the addition of damping increase or reduce the vibration? It'll increase it. But damping's a necessary evil. You need some damping in the system. So if you bump it, it doesn't sit there and oscillate all day long.

Next. OK, this is a platform. Do you think I could actually do it? Did you read this?

So this is a Coast Guard light station off of Cuttyhunk down off of Woods Hole-- basically, I was doing this-- in time with the motion of the platform. Resonance is a wonderful thing. If you can make the force be right at the natural frequency of the structure, it actually doesn't take a lot of force to drive the amplitude to pretty large amplitudes, if the damping is small.

So I think the damping in this cases is about 1%. And that means the amplification, the dynamic amplification 1 over 2 zeta is about 50. So I actually could do this. This is a true story. OK, next.

For small motions about the horizontal, you expect the natural frequency to be a function of gravity. So this is, oh, some of you, about equal, yes no. But it's just horizontal, the torque that gravity provides is some Mg pulling down on the center of mass somewhere in that body. Not at the pivot, but let's say some distance A away.

So the torque, the gravity, the restoring torque is some Mg cross r cross Mg, Mgr if r is the distance. And that's the torque. And that length of that moment arm might vary. It's going to vary like cosine theta.

Around horizontal line, if theta is what's, for small angles, what's cosine? It goes to 1. So you find out that this just looks like Mgr. It's for small angle of vibration. And you can in fact get it out of-- it doesn't enter into the equation for the natural frequency.

So the natural frequency of a thing won't be a function of gravity because of this small angle vibration around a horizontal point. OK. One, when the acceleration of the system is one half that required to make the mass slide, what's the magnitude of the friction force?

So friction is one of those things that is only as big as you need it to be.

So even the largest friction that this thing can sustain is in fact mu mg-- answer A. But f equals ma if the acceleration is half of what is required to have that thing be just slip. It will just slip when you are at a force which is mu mg. And so that force is equal to mass times acceleration.

The acceleration then you can figure out what that will be just when it slips. But now if you reduce acceleration to half that, the friction force required to keep it in place is only half as big. And it will be that friction force and it will be half of mu mg. So it's actually B.

And next-- is that it? No. OK. This is a simple but actually sometimes hard to see through question. What initial conditions will be required? This problem can be solved by initial conditions.

This mortar launches its shell. And the trick to this question, the key to this question is for your mathematical model of the system-- your equation in motion-- is write the equation of motion without the shell. Because once it shoots this thing, the shell's gone. And it's vibrating. It's now a system without that 25 kilogram mortar shell part of the system. It's gone. Now it's just the mass of the system without the mortar shell.

And there are two initial conditions that then you can say when you shot the mortar shell, that was a certain amount of momentum. And from conservation of momentum you can figure out what the momentum of the main mass has to be-- equal and opposite to the shell you shot. So that gives you an initial velocity. But there's also an initial displacement in this problem. So that's the key to figuring this out.

So what initial conditions would be required? And it's C-- both an initial velocity and an initial displacement. But the key is to make your mathematical model about the system without the shell. OK? Good. Is that it? All right.

So today we're going to pick up where Professor Gossard left off. But I'm also going to do a bit of a summary right now about vibration and modeling the different kinds of systems that we talk about when we talk about vibration. They vary from simple, single degree of freedom oscillators, like a simple pendulum-- one degree of freedom-- to continuous systems-- beams and vibrate.

So I'm going to try to give you just sort of an overview of vibration just to sort of give you a little map of information. Kind of to let you know what the body of vibration analysis is and what part of it we're covering in this course. So I think I will use a little more board.

So we classify dynamics problems into, for convenience, rigid bodies-- rigid body dynamics and flexible bodies. One way to think of it. And this course is basically about rigid body dynamics. And under this we then have two categories that are convenient-- single degree of freedom systems and multiple degree of freedom systems. For the purposes of vendors-- talking about vibration.

Single degree of freedom systems have one equation of motion. And if they vibrate, they have-- and if I'll put over here-- if vibration occurs then you have one natural frequency. And it's sort of silly to talk about a mode shape for a single degree of freedom system, because it's only relative to itself. So one natural frequency and one sort of degenerate mode shape.

Multiple degree of freedom systems have n equations of motion for the number of degrees of freedom. And if they vibrate they have n omega i's, or m values of omega i for i equals 1 to n. You get n natural frequencies of the system. And you will get with it n mode shapes. So a n degree of freedom, this is equal to the number of degrees of freedom. An n degree of freedom system will have n natural frequencies and n mode shapes that go along with it.

Now, what about what about flexible bodies? So a taut string like a guitar string. And actually I should say over here these rigid body things-- we have found what kind of equations of motion? These are ordinary differential equations. And there's a finite number of them and so forth.

The flexible bodies like taut strings are described by partial differential equations. The number of degrees of freedom n here is the number of degrees of freedom actually goes to infinity. And you get an infinite number of omega i's, the natural frequencies, and an infinite number of corresponding mode shapes.

So just about everything in the world can be made to vibrate. So how do you tell if a-- you've got a mechanical system, rigid bodies, you've got three degrees of freedom. How do you know whether or not it's going to vibrate? It will exhibit vibration?

Well, one thing you could do is figure out all the equations of motion and solve them and see if cosine omega t is a solution. Right? That's the hard way. The other way is to go up to it and give it a smack and see if it vibrates. That's the simple way.

If you have the mechanical system just give it a whack. And if it oscillates around some stable equilibrium position, it exhibits vibration. So this is a flexible system.

You can actually probably see this from there. Just by giving this frame a smack, it will sit and vibrate. And it does it at some natural frequency. But that's a continuous system.

This continuously improving little demo-- so Professor Gossard for his lecture last weekend had done this really neat embellishment, which allows you to figure out and excite the two different natural modes. But this system you have equations of motion for it. You could write it. And if you come up and give it a whack, it oscillates. And you could also find out that sure enough cosine omega t sine omega t are solutions to the equations of motion.

So systems that vibrate are systems that oscillate about static equilibrium positions. And another way you can say that is when mechanical vibration occurs, there's always an exchange of energy between kinetic and potential, kinetic and potential. So our pendulum it goes-- when it reaches zero velocity up here, it's all potential energy.

It reaches maximum velocity down here, it's all kinetic. And it goes back and forth. As it sloshes back and forth the energy system from kinetic to potential and back again. All vibration has that property. So that sets some basic properties of vibration.

And now there's a whole body of knowledge about vibration. And we choose, or for the purposes of this class, we choose to break it down into two kinds of vibration. And one is what we call free vibration. And that we've learned already is response, only a response. It's a response to initial conditions and what we call forced vibrations.

Now, forces can come of all kinds. And for the purposes of this course we look at a particular kind of force. So we focus on harmonic excitation. So excitation that is of the form cosine omega t, or e to the i omega t. These are external excitations.

So we choose to break down the analysis of the vibration of systems into response to initial conditions called free vibration, no external forces, and force vibration. But we focus on a particular kind-- harmonic. And we go even one step further and say we're only going to study steady state.

And steady state means you've waited a long time. Turned it on, let it shake for quite awhile. All the initial startup transients have been damped out. And you're left with a steady state vibration. And that leads to things like the transfer functions for single degree of freedom systems that we've talked about.

Now, there's one other breakdown or subdivision that we need to talk about. And that is whether systems are linear or non-linear. And this is all set up so you can see it. This is a double pendulum. How many degrees of freedom? Two. And in general, do you think the equations of motion of this thing are going to be non-linear? Right.

Just a simple pendulum is the restoring torque is Mgl sine theta. So you know it's got sine theta and that. And this one gets quite messy. And especially if you give it large amplitudes. And that really isn't vibration. It's not. It's looping all over itself and then doing other things. So cosine omega t is not a solution. It's not a solution to this. It's got to be more complicated than that.

So when this thing is exhibiting large motions, the equations of motion are completely non-linear. And you're going to need a computer to crank out the full solution to integrate these non-linear equations of motion. But as the amplitude settles down to something pretty small, now it's vibrating about an equilibrium position. The equilibrium position is straight down. And the damping of it has made it such that the only motion left is what's called its first mode of vibration.

And so if we linearize the equations of motion, assuming small amplitudes around static equilibrium positions, then we can find a vibration solution and work it out by hand probably. That's first mode for this system. And if I'm careful-- there's second mode. And for small oscillations it has a very clear single frequency that it vibrates at.

The amplitude decays over time because of damping. And for every natural frequency there is a particular mode shape that goes along with that natural frequency. The first one for this system-- I have to wait for this thing to damp out. It's got a little mix of the two, but as it-- the second natural frequency motion dies out faster than the first because it has more cycles per unit time.

So it settles down. This is now mostly first mode vibration. And you can see that both move in the same direction, the bottom one a little more than the top. And that's the first mode. It has a unique natural frequency. And a mode shape that is specific that goes along with that natural frequency.

So this further break down here, I'll call it, is basically into non-linear and linearized. So in our discussions of vibration in this course we basically only talk about this. So we're only doing-- so that's quite a breakdown. You start at all possible vibration systems, rigid bodies, single degree of freedom, multidegree of freedom, finite number of degrees of freedom or continuous.

They can have linear or non-linear equations of motion. But if we require them to be linear, and that's what we're going to look at then you sort of narrow this down what we're looking at to this. So there's lots of other things possible to look at it, like that really non-linear motion of that two dimensional thing. But our study of vibration is here.

So this is what we're doing in 2003. But there's a lot of important problems that are covered by that. Lots of real things in nature that are problematic for engineers and problematic for design can be analyzed with linear equations of motion.

And even if they're not linear, if you do the linear solution first, it gives you a starting point to think about what's the behavior of the non-linear systems. But this is our study of vibration. And we're going to do that for-- and what we had started doing-- in two ways. We look at the response to initial conditions called free vibration. And we look at response, steady state response of now these linear systems to force vibration.

And last week you were looking for the first time at Professor Gossard's lecture about the free vibration response of basically a two degree freedom system. So why do two degree of freedom systems? Well, it's the simplest next step up from a single degree. And they're sort of mathematically tractable. You can do them on paper.

We emphasize looking at two degree freedom systems because we can do the math on the board, do the math on the paper. But as you get to more degrees of freedom, you basically are going to have to do-- it's easier to do it using the computer. And in order to do that you need to know some linear algebra.

So I'm kind curious. In terms of linear algebra, like multiplying two matrices together, or finding the determinant of a matrix, or inverting a matrix, how many of you actually have been taught that? How may perhaps have it? Do you do that in 1803 now? Is that where you do it?

OK. Good. So that's helpful. I wasn't sure whether or not. I can assume that you at least know what the determinant of a matrix is. That's great. That's really helpful.

OK. Let's talk. So we want linear equations of motion. And I've done a little bit about linearization but not much. So let's talk a little bit about that for a second. For a pendulum we know the equation of motion for it. And actually we could make this a more complicated pendulum. It could be a stick or any rigid body swinging about this point A.

We know that we can write the equation of motion Izz with respect to A, theta double dot, plus Mgl-- omega L. The distance here to wherever g is, Mgl sine theta. And for free vibration that's all there is to it.

And to linearize this equation we just say, well, we know that sine theta is equal to theta minus theta cubed over 3 factorial plus theta to the fifth over 5 factorial. And the cosine theta-- just to have it available here-- is 1 minus theta squared over 2 factorial and so forth.

And when we say linearize, we really mean we want our equations to involve the motion variables at most to first order. So the first order term for sine is theta. There's no first order term for cosine. Theta squared is non-linear. So the small angle approximation to cosine is it's approximately 1. And to sine is it's approximately theta for small motions. Theta-- small.

So when we linearize this equation, we just substitute in for sine theta its linear approximation. And we get Mgl theta. So we've seen that one many times. And that's your linearized equation of motion. But on this week's homework you've got a harder problem. And that's our cart.

And here you have a theta and an x are your two equations. And you've worked this problem before. And you know with the previous homeworks you've gotten the equation of motion. I'll write one of them down here.

So one of the equations of motion is-- this is m1, m1, k, b. And this is a stick, it's l long, g in the middle. So the equation of motion for this looks like m1 plus m2, x double dot plus m2 l over 2, theta double dot, minus m2 l over 2 theta dot squared theta, and then plus bx dot, plus kx, and equals, and in fact, this one has a force on it. It's equal to F of t.

Now, is that a non-linear equation? So this is the force equation mass times acceleration or forces. You know you've got another equation of motion in here, which is the torque one. This is just one of them. So is it linear or non-linear? How many think it's non-linear?

OK. If I have number the terms here-- one, two, three, four, five. And that's not a motion. This doesn't involve motion. So if one through five, which ones have a non-linear term?

AUDIENCE: Three.

J. KIM VANDIVER: Three. OK. So how do you linearize that thing?

AUDIENCE: You make it zero, because it's second order.

J. KIM VANDIVER: In fact, it's third order. So the reason I wanted to mention this today-- if you haven't thought about being confronted with linearization problems before-- we're trying to linearize the system so that we can by making it linear we can make cosine omega t a solution. Right? We want cosine omega t to be a solution to this thing.

So if you let-- so you've got a term that looks like ml over m2l over 2, theta dot squared theta. Well, theta in this problem of some function of time we're hoping-- we want to find a solution that has some amplitude times, say, a cosine omega t. And theta dot is minus omega theta naught sine omega t.

And so that expression up there, the magnitude of that expression or the magnitude of theta dot squared theta is proportional to-- you get a theta naught here and you get omega squared theta naught squared here. So this term is proportional to theta naught cubed. And if the angle theta is small, then a small angle cubed is really small.

And so the way you linearize this equation is to throw this out. So when you've done all your tricks you can, like replacing sine theta with theta and cosine theta with one, and you still end up with terms have a higher order than one in the motion variable, theta or x, you throw it out. So if you throw that term out then you end up with a nice linear equation in motion.

OK. So now for the rest of today we're going to talk about free vibration solution. So we're not going to worry for the moment about the force vibration steady state transfer function stuff. We're talking just about free vibration. And this is of linear equations of motion.

So vibration is a pretty big body of knowledge. And we're doing an introduction to vibration in about half a dozen lectures here. So there's lots of things that I'm not going to have time to teach you, but there are a few things I really want you to go away with and understanding.

And one of these key concepts is that the vibration of a multiple degree of freedom system-- say this is a two degree of freedom system. That the vibration of this system, the free vibration, can be made up of the sum of two parts at any vibration of this system at all. So at any arbitrary set of initial conditions I give it-- I let it go.

The key concept is the response will be made up of two pieces-- vibration in each of the two modes. And if you can solve the vibration that's in first mode-- first mode is the one where they're going kind of together, second mode they're opposite one another. That the total solution can be made up of a contribution for mode one and a contribution from mode two.

So this is this concept called mode superposition. It's really quite powerful. So you can figure out the response of the first mode in the system, figure out the response of the second mode's contribution, add them together, and that's the total solution.

And this concept works-- there's all sorts of caveats that one gets into-- but basically this is true for all lightly damped systems. You get into heavy damping and strange damping, you have to make some adjustments. But for lightly damped systems you'll find that this concept of mode superposition works out just fine.

So an illustration of this, a really simple illustration-- in some ways easier than this one. I don't know if I can get this where you can see it in the picture or not. Maybe not really. This is just two little lead weights. This is a double pendulum. It has two natural frequencies. One is that one.

You can see the two weights going the same direction. The bottom weight at a little bit more larger angle than the top weight. And it's at a particular frequency. And that's the mode shape that goes through this frequency.

So another key concept is that for free vibration the total solution is made up of the free vibration of each mode. And each mode has a particular frequency and a particular shape to it. So that's the first mode frequency and the first mode shape.

The second mode-- I have a little harder time getting it started-- it looks like that. Masses move in opposite directions. It's kind of rotating around where you can't see it. I have to do it in the plane. It's hard to do here. It doesn't want to behave like it's confined to a plane. They're going opposite directions. And the frequency is higher.

But this motion, that mode shape, is a fixed feature of this mode of vibration along with this natural frequency. So this idea of mode superposition-- and a second concept here is that for free vibration of each mode it oscillates at a unique frequency for this two degree of freedom system. You have two natural frequencies-- omega 1 and omega 2. And at each omega n there is a corresponding mode shape.

So any vibration of a linear system, free vibration of it, any vibration at all is composed of a superposition of the two modes. Part of this motion is in the first mode at its natural frequency and in its shape. And part of the motion has a second contribution, which is at the natural frequency of the second mode and in its shape.

So I'm going to give you a quick demo and ask you-- let's if you can use what I just said to analyze a motion. So this is just a block on some strings. And I'm going to show you a motion. And I want you to tell me whether or not it could possibly be a natural frequency motion in one mode, or the other answer is it's a sum of multiple modes.

But I'm going to show you a motion, and I want you to tell me and argue on the basis of what I've just told you whether or not you are seeing a single mode of vibration. And maybe I'll use the clamp here so I don't have to stand there and hold it. I'm going to just place this.

So the way you do free vibration is you give it an initial displacement, some initial conditions, and let go. So I'm going to pull this over and back and let go. And just watch closely what you see it do.

All right, now it's doing more of what I want. It's like it's going in a circle right now. And now it looks like it's just going back and forth on a diagonal. And then it's going to start circling the other way. It's going in a circle. And now it goes to on the diagonal-- left and right. And then it starts back into a circle again.

Are you observing a natural mode of vibration? It looks like it's single frequency, right? This looks like it's all happening at one frequency. But is it a natural mode, a unique natural mode? Who wants to make a case for whether it is or isn't? How many believe that you're seeing a natural mode of vibration? None.

How many think you're not seeing a natural mode of vibration? Let's see if you're awake. OK. So you don't believe that it's natural mode. Make the case. Why? How do you use sort of this definition of a natural mode to tell me why this can't be?

AUDIENCE: It looks like a superposition of at least two different kinds of vibration.

J. KIM VANDIVER: OK. The evidence that you see is because what does it do?

AUDIENCE: It circled sometimes. And sometimes it goes straight back and forth on a diagonal.

J. KIM VANDIVER: OK. So it circled around part of the time and then goes straight back and forth part of the time. Is it a constant mode shape of vibration? No. And that's all you need to observe. If the thing doesn't keep a constant single shape at a single frequency, it's not a natural mode.

So let's do a different case. I'll deflect it just this way. And ignore that little bit of torsion. So it's just going back and forth in line. Other than slowly damping out, that has just one motion to it, and it's at one natural frequency.

So do you think that's a mode? That probably is two. And so is this one. And ignore that high frequency. Now it's just back and forth. It's just a pendulum. And it just stays in just pendular motion, no circling around or any of that. So that's also natural, and it occurs at a particular frequency.

So these are two individual pendular emotions-- one this way and one that way. And what I was doing at the beginning is I pulled it to the side, which would start one of those modes. And I pulled it back, which will put some energy into the other mode, and let it go. And now what you have is the sum of these two different motions adding up. It goes in circles and then in straight lines. And the fact that they-- this is a phenomenon called beading.

And it is because these two pendulums, even though they have strings of the same length, they actually have slightly different natural frequencies. They're each single degree of freedom systems. They're two independent single degree of freedom systems, each with their own natural frequency. But if you mix them then they're going to exhibit this motion.

So that's something really important to remember. A quiz question that I like to ask is-- it's easy to grade and it's no math required-- is to literally-- I've often done this in exams-- walk in with something like that block of wood and say, is this a natural mode?

Time to do one-- let me see here. So now let's pick up where Professor Gossard left off. Let's go talking about natural frequencies and mode shapes of linearized two degree of freedom systems. But I want to generalize a little bit on what he did.

So he, in his lecture, analyzed this system like this. I'll just kind of put the highlights here. And this is now solving for natural frequencies and mode shapes. He came up with a set of equations of motion for this. This was, I guess, M1, M2. And the equations of motion for this are m1 in matrix form.

Now I'm going to do this to emphasize something. In general there could be damping in our linearized system. And we have a stiffness matrix-- K1 plus K2 minus K2. And in general there could be forces, which are functions of time on that system.

Now, if we want to find natural frequencies in mode shapes, we go looking for what we call with the undamped natural frequencies in mode shapes. So this problem doesn't even have dampers in it. But if it did for the purpose of finding natural frequencies in mode shapes, you just set to 0. And with the forces you do the same thing.

And now you have undamped, unforced equations of motion. And this is then of the form of mass matrix times an acceleration vector, X1, X2, plus a stiffness matrix, times a displacement vector equals 0. So in matrix notation it looks like that.

This is the way you would do any rigid body vibration problem. This is two degrees of freedom. But this is the general expression for an n degree of freedom system. If we had three masses here, then these would be 3 by 3 matrices instead of 2 by 2's. So that's the basic formulation.

And you went through last time with 2 by 2. You can actually go through and find the fourth order equation in omega and solve for two roots of omega squared. And you've got the two natural frequencies, plug them back in. You've got the two mode shapes that go along with them. So you did it that way by hand so you can see how you can work out the natural frequencies.

How can you do-- I'm going to show an approach that you'd more likely use on a computer. And if you get the larger order n degree of freedom systems, you're going to want to do this-- instead of by hand-- have a computer do the work for you. So this is the generic form. And let's just assume for a minute it's an n degree of freedom system.

So these are n by n matrices. How would we find the natural frequencies and mode shapes of this general system? So you assume solutions of the form of what I've been describing-- a natural mode. Any natural mode of the system has a particular shape to it and a particular frequency. And that's the key.

That's the key assumption here. You assume solutions of the form that this vector x-- instead of writing it in brackets like this, I'm going to make this. So X here is just with a line underneath it. So x is of the form X1 of t down to Xn if you have n degrees of freedom. You're looking for a solution for that thing. And it's going to have an amplitude to it-- A1 down to An. And this is any one mode.

So any one mode will look like a set of amplitudes that govern its mode shape. And it will oscillate. We can write the oscillation as cosine omega. And I'll put an i here, it's the i-th natural frequency minus some phase angle.

So in general, each mode-- assume solutions-- I'll say here for each mode. So each mode, any mode, mode I will look like this. It will have a shape to it governed by this. And these are basically on constants. Once determined, this is just a constant.

And here's your time dependence. And it's going to-- I left out my t-- oscillate at some natural frequency. So we know this is what the solution has to look like. And we can take this and plug it in to this equation.

This vector of responses is some vector of amplitude times the cosine omega t minus phi. And just plug that into this set of matrix equations. Note that x double dot is just a-- you get minus omega squared a cosine omega t. And we now substitute these into here.

You get minus omega squared for the first term. Minus omega squared times the mass matrix a cosine omega t minus phi, plus this stiffness matrix a-- better consistent notation here, excuse me-- a cosine omega t minus phi. And all that's equal to zero.

So these go away. You can cancel them out. And we can factor out this a quantity. And we have minus omega squared m, plus k times a equals 0. So you can do this with any linearized n degree of freedom system that you know has a vibration solution to it. These are the unknown mode shapes.

And so in order to satisfy this equation, either this a has to be 0, which is a trivial solution. There's no motion, no mode shape-- or which this is trivial, not too useful. Either a has to be 0, or the determinant of this quantity has to be 0. But the way you do that on a computer-- so that would be beginning the way you would analyze this by hand.

You find the determinant of that matrix. And if it is a two degree of freedom system, you'll get an equation n omega to the fourth, which has two roots for omega squared. If it's a three degree of freedom system, you'll get an equation of omega to the sixth when you write out that determinant. And it has three roots for omega squared.

An n degree of freedom system has an equation that's of order 2n omega to the 2nth power. And it'll have n solutions or roots for the natural frequency for omega squared. But that would be if you're trying to grind this out by hand. The way you do this on a computer-- maybe I can get a little bit more on here. Come back here.

So I'll go back to the earlier form I had here plus ka equals 0. And I'm going to multiply by m inverse. So if I invert the mass matrix, if I multiply a matrix by its inverse, what do you get? So if I multiply m times m inverse?

AUDIENCE: A unit matrix.

J. KIM VANDIVER: You get a unit matrix, right? It has ones on the diagonal. So I'm going to multiply through here by this. And so this gives me a minus omega squared. And m times m inverse gives me the unit matrix-- ones on the diagonal. Times a plus M inverse times ka equals 0. And this product is just a matrix product-- m inverse times k. And I'm going to call it the a matrix.

And I'm going to move this to the other side. So I have a linear algebraic expression of the form a times the vector equals omega squared times the unit matrix times a. And I could go ahead and multiply this out. For example, this times that and I'll get a vector.

So this just looks like omega squared a if you multiply it out. The vector times the matrix is a vector on the left side. A vector times a matrix gives me back a vector. It's just the unit matrix. So it gives me back the vector times omega squared. This is in what is known as standard eigenvalue formulation.

It's a standard eigenvalue problem now. It's a problem of the form a times a vector equals something lambda times a. A parameter which we know happens to be the frequency squared. But this is standard eigenvalue formulation. Yeah?

AUDIENCE: I was just asking if you wrote down omega squared a because it's equal to the length up there.

J. KIM VANDIVER: Ah, good. Omega squared a. So a times the unit vector you get a back as a vector. And I got the omega squared in front of it. And oftentimes in a manual for Matlab or something they'll describe this as some parameter.

It's a constant times a. And this is standard what they call eigenvalue formulation. And in Matlab if you say, for example, E equals EIG of A. This returns a vector, which is the natural frequency squared. It will return these lambdas.

And the first one is omega 1 squared down to omega n squared. And if you go with this function, if you go a little further, if you say V comma D is EIG A, then this gives you two matrices back. It gives you V. And V is a matrix, which its columns are the mode shape.

So A1 to An, this is mode 1 over to A1 to An for mode n. It gives you two matrices. One that's that. And another one a D matrix, which has the lambdas-- lambda 1 lambda n on the diagonals. And it's a diagonal matrix.

So it gives you two matrices back. One that has the eigenvectors, the mode shapes. And another matrix whose diagonal elements are the natural frequency squared. And that's all there is to it if you do this numerically.

And there's lots of different programs. There's multiple ways of doing this in Matlab. When you do it this way it doesn't come out sometimes nicely ordered and what I call normalized. But it does produce the eigenvalues.

They're called eigenvalues and eigenvectors. The eigenvalues are the lambdas, the natural frequencies squared. And the eigenvectors are these mode shapes that go with each natural frequency.

Once you know the natural frequencies and mode shapes, now we want to get back to talking about solutions. This idea of mode superposition. And if you give it a set of initial conditions, what is the response? How do you add these two modes together?

So let's go back now. We'll return to two degree of freedom systems like this one to do an example. And we assume that the solution was sum A1, A2, cosine omega t minus a phase angle. That each mode would have this character to it.

And I'm going to normalize my mode shapes. So for each mode shape of the system-- so this could be for mode one. This is the mode shape for mode one. This is natural frequency one and phase angle one.

Each mode shape-- I could write this then as-- I could factor out the A1. Just pull out A1, divide each member by A1. So this can be written as A1 times 1 and A2 over

A1. So I've just factored out.

So for mode one it's normalized mode shape-- by normalize you just pick some way that you repeatedly use, consistent in its use. I often say let's make the top element of the vector 1. And to make the top one 1 you factor out whatever its value is that you get back from the computer or from your calculation.

You factor that out of every member. Now you have a normalized mode shape whose top element is 1. There's lots of other normalization schemes. That's just one way to do it. And that's one of the mode shapes.

The total solution is X1, X2-- and this is where the mode superposition part comes in-- is some undetermined constant A1 times the mode shape A2 over A1 for mode 1 cosine omega 1 t minus phi 1. And I'm going to run out of room.

And now the responses to initial conditions-- this has got another term. I'm just going to rewrite it here. So we're looking at-- our total motion response now by mode superposition will be A1, 1 A2 over A1, mode 1.

So the free vibration response of any two degree of freedom system, linearized equation, any two degree of freedom linear system can be made up of the sum of two terms. The motion at its first natural frequency in its first mode shape. And another term is the motion at a second natural frequency and its second mode shape.

But now you have two undetermined constants out here-- A1 and A2. Where do they come from? You have to use your initial conditions to get those. I'll write down-- let's see. I have just maybe enough time to write this down. These are functions of time. So A1 and A2 come from the ICs, the initial conditions.

So at t equals 0, for example, plug in t equals 0 into here. You get cosine phi. And over here another phi 1 and another way over here. Cosine of minus phi 1 is cosine phi 1. So if you put in t equals 0, you find out the X1 at 0, which I'll write X1 0, and X2 of 0 is equal to-- I'll actually write them out. This is going to be A1 times 1 cosine phi 1, plus A2 times 1 cosine phi 2.

And the second equation that this gives you is A1-- and now to keep from writing these many, many times, I'm going to let this first A2 over A1 for mode 1 be R1, and the second one A2 over A1 for mode 2. I'll just call it R2. And then I can write this out.

So the second equation, this is R1 cosine phi 1, plus A2 R2 cosine phi 2. And now I have initial conditions that are normally given. This is an initial displacement on 1 and a initial displacement on 2. The phis I don't know. And the A's I don't know. I have four unknowns and two equations.

How do I get two more equations? I take the derivative of this expression to get velocity. And I get an A omega here and an A omega here. And I plug in t equals 0. And I get two more equations. So X1 dot and X2 dot equal at t equals 0. This gives me two more equations.

And if I have a place to write them-- for example, X1 dot-- or X1 0 dot, the initial condition on velocity. X2 dot-- this is two equations. I've only got time to write down one of them. And you could do the other for exercise. You find this is A1 omega 1 sine phi 1, plus A2 omega 2 sine phi 2. And the second equation is you get A1 R1 omega 1 sine phi 1, plus A2 R2 omega 2 sine phi 2.

So now you have one, two. And this is the initial conditions on velocity. These are initial values of velocity. Now you have one, two, three, four equations and one, two, three, four unknowns-- the A1, A2, phi 1, phi 2. And I guess I will give you the answer, so you have it once.

So a little tedious but this is sort of in the spirit of we do two degree of freedom systems so that we can see how it works. And then for larger degrees of freedom systems you would do this with a computer. But the solution for A1 is 1 over R2 minus R1. It's all in terms now of things you know.

These R2's and R1's are part of the mode shape, and the rest is initial conditions. R2 x1 0, minus X2 0 squared, plus R2 V1 0, minus V2 0 quantity squared, over omega 1 squared, the whole thing square root. But now this is all stuff you know.

The given initial conditions on velocity, given initial conditions on displacement. You know the natural frequency. You know the pieces of the mode shapes. Just plug it in, you're going to get a number.

The magnitude of A1 for the given initial conditions. A2-- a very similar expression-- minus R1 X1 0 plus x2 0 squared And you'd solve for A2 and phi 1-- it's a little simpler-- minus tangent inverse, phi 2 0 minus R2 V1 0, all over omega 1 R2 x1 0, minus x2 0. That's one of the phase angles. And the other phase angle, a very similar expression. Minus tangent inverse, minus R1 V1 0, plus V2 0 omega 2 R1 x1 0, minus x2 0.

So just expressions in terms of the initial conditions and you can get all four quantities. You can also do this on the computer. But in the few short lectures that we have we're not going to get into that. But this just shows you where it goes. You could do this now. There are straightforward ways of doing it with matrix algebra on the computer.

Next time I'll do maybe just a quick example-- I didn't quite get to it today-- of a response to initial conditions problem. Plug it in there. See what happens. But we're out of time. See you on Thursday.