2.003J/1.053J Dynamics and Control I, Spring 2007
Professor Peacock
5/7/2007

Lecture 21

Vibrations: Second Order Systems - Forced Response

Governing Equation

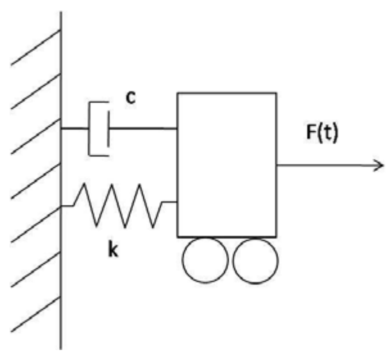

Figure 1: Cart attached to spring and dashpot subject to force, $F(t)$. Figure by MIT OCW.

$$m\ddot{x} + c\dot{x} + kx = F(t)$$
$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{F(t)}{m} \quad (1)$$

ζ: Damping Ratio
ω_n: Natural Frequency

Forced Response - Particular Solution $x_p(t)$

Can use Fourier Series or Laplace Transforms

Start with a simple case $F(t) = f =$constant

$F(t)$ is constant

The complementary solution below requires $\zeta < 1$.

$$x_c = Ce^{-\zeta\omega_n t}\cos(\omega_d - \phi)$$

Subscript c for complementary solution.

$x_p = ?$

Try $x = At + B$.

$$2\zeta\omega_n A + \omega_n^2(At + B) = \frac{f}{m}$$

$A = 0$, $B = \frac{f}{m\omega_n^2} = \frac{f}{k}$
$\left(\omega_n = \sqrt{\frac{k}{m}}\right)$

Therefore:

$$x = Ce^{-\zeta\omega_n t}\cos(\omega_d t - \phi) + \frac{f}{k}$$

$x_c = Ce^{(-\zeta\omega_n t)}\cos(\omega_d t - \phi)$: unknown constants set by initial conditions
$x_p = \frac{f}{k}$: determined by forcing; independent of initial conditions

Calculating C and ϕ

$$x(0) = C\cos(-\phi) + \frac{f}{k} = 0 \tag{2}$$

$$\dot{x}(0) = -\zeta\omega_n C\cos(-\phi) + C\omega_d\sin\phi = 0 \tag{3}$$

The example initial conditions are $x(0) = 0$, $\dot{x}(0) = 0$

Equation (3) gives $\tan(\phi) = \frac{\zeta\omega_n}{\omega_d}$.

$$\frac{1}{\cos^2\phi} = 1 + \tan^2(\phi) = 1 + \frac{\zeta\omega_n^2}{\omega_d^2} = \frac{\omega_d^2 + \zeta^2\omega_n^2}{\omega_d^2} = \frac{(1-\zeta^2)\omega_n^2 + \zeta^2\omega_n^2}{(1-\zeta^2)\omega_n^2} = \frac{1}{1-\zeta^2}$$

$$C = -\frac{f}{k}\frac{1}{\sqrt{1-\zeta^2}}$$

Complete Solution

$$x = \frac{f}{k}\left[1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\cos(\omega_d t - \phi)\right]$$

As $t \to \infty$, $x \to \frac{f}{k} = x_p$

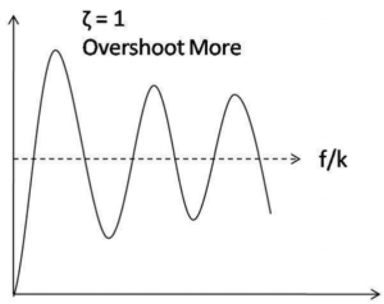

Figure 2: Solution to differential equation. Figure by MIT OCW.

What actually happens is set by ζ and ω_n.

x_p can be thought of as the steady state response once the transients die down.

So we will now focus on the steady state response. Of particular interest is the frequency response (i.e. response amplitude and phase as a function of forcing frequency.

$F(t)$ is a periodic function

$$m\ddot{x} + c\dot{x} + kx = F_0 \cos\omega t \tag{4}$$

$$\frac{d}{dt}(T+V) = (m\ddot{x} + kx)\dot{x} = (F(t) - C\dot{x})\dot{x}$$

In steady state $< F(t)\cdot\dot{x} >=< c\dot{x}^2 >$.
$x_p =$? Could choose sine and cosine, but use complex exponentials. Easier to work with phases.

Convenient to write and solve for:

$$F = \mathrm{Re}\left\{F_0 e^{i\omega t}\right\}$$
$$x_p = \mathrm{Re}\left\{\mathbb{X} e^{i\omega t}\right\}$$

$\mathbb{X}$ is a complex number. Substitute in Equation (4).

$$(-m\omega^2 + ci\omega + k)\mathbb{X}e^{(i\omega t)} = F_0 e^{(i\omega t)}$$

$$\mathbb{X} = \frac{F_0}{k - m\omega^2 + ic\omega} = \frac{F_0/k}{\left[1 - \frac{\omega^2}{\omega_n^2} + 2i\zeta\frac{\omega}{\omega_n}\right]}$$

$$\mathbb{X} = |\mathbb{X}|e^{-i\phi}$$

$|\mathbb{X}|$: Amplitude
$e^{-i\phi}$: In phase or out of phase?

With complex numbers, bring complex part to numerator instead of denominator. Multiply by complex conjugate.

$$1 - \frac{\omega^2}{\omega_n^2} - 2i\zeta\frac{\omega}{\omega_n}$$

$$\mathbb{X} = |\mathbb{X}|e^{-i\phi} = \frac{F_0}{k} \cdot \frac{1 - \frac{\omega^2}{\omega_n^2} - 2i\zeta\frac{\omega}{\omega_n}}{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}$$

$$x_p(t) = \mathrm{Re}\left\{\mathbb{X}e^{i\omega t}\right\} = \mathrm{Re}\left\{|\mathbb{X}|e^{-i\phi}e^{i\omega t}\right\} = \mathbb{X}\cos(\omega t - \phi)$$

$$\mathbb{X} = \frac{F_0}{k} \cdot \frac{\sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}}{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}$$

$$\boxed{\mathbb{X} = \frac{F_0}{k} \cdot \frac{1}{\sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}}}$$

$\phi =$? Ratio of real and imaginary parts.

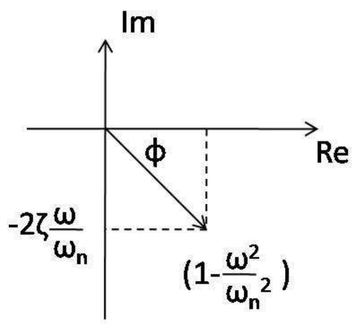

Figure 3: Determining ϕ using the real and imaginary parts of the solution. Figure by MIT OCW.

This diagram corresponds to $e^{-i\phi}$.

$$\boxed{\tan\phi = \frac{2\zeta\frac{\omega}{\omega_n}}{1-\frac{\omega^2}{\omega_n^2}}}$$

Analysis For $\omega \to 0$

(Forcing Frequency $\to 0$)
System acts as if it is at steady state.

$|\mathbb{X}| = \frac{F_0}{k}$, $\phi = 0$ or π. $\phi = \pi$ is not physically meaningful.

Analysis For $\omega \to \infty$

If one forces the system too fast, system cannot respond.
$|\mathbb{X}| \to 0$, $\lim_{\omega\to\infty} \tan\phi = 0$

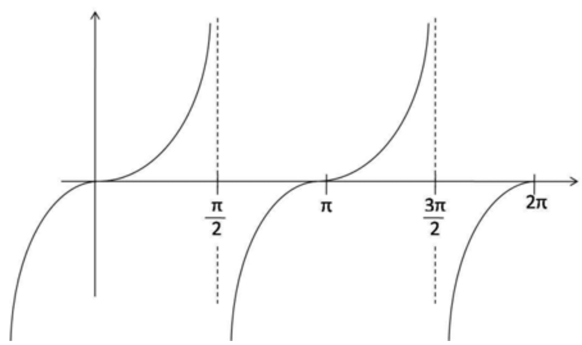

Figure 4: $\phi = \pi$. Approaching 0 from a negative number so $\phi = \pi$. System is completely out of phase. Cart moves in opposite direction from forcing. Figure by MIT OCW.

Analysis For $\omega = \omega_n$

$$|\mathbb{X}| = \frac{F_0/k}{2\zeta} = \frac{\mathbb{X}_{\text{static}}}{2\zeta}$$

Also true for $\zeta << 1$.

$\phi \to \frac{\pi}{2}$. We start at $\phi = 0$, then we approach $\tan\phi \to \infty$ so $\phi \to \frac{\pi}{2}$.

$$(-m\omega_n^2 + ic\omega_n + k)\mathbb{X}e^{i\omega t} = F_0 e^{i\omega t}$$

$-m\omega_n^2 + k = 0$

Just phase shift and damping:

$$(ic\omega_n)\mathbb{X}e^{i\omega t} = F_0 e^{i\omega t}$$

The maximum frequency response is not necessarily the natural frequency response. To find maximum frequency response, differentiate.

$$\frac{d}{d\omega}\left[\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2\right] = 0$$

Minimum of denominator $\Rightarrow$ max$|\mathbb{X}|$ $\Rightarrow \omega_{max} = \omega_n\sqrt{1-2\zeta^2} \leq \omega_n$. $0 < \zeta \leq \frac{\sqrt{2}}{2}$. Notice ω_{max} is less than ω_n.

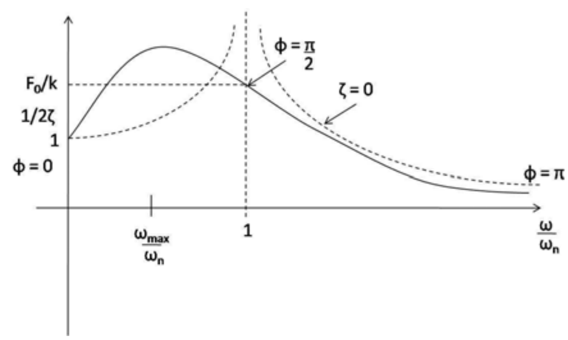

Figure 5: Summary graph of $\mathbb{X}$ vs. (ω/ω_n) for forced response. $\mathbb{X}$ starts out at 1 when (ω/ω_n) equals zero, and ϕ equals 0. Then X goes through a maximum at (ω_{max}/ω_n), which is less than 1. At (ω/ω_n) equals 1, ϕ equals $\pi/2$, $\mathbb{X}$ equals F_0/k. $\mathbb{X}$ continues to diminish and approaches zero for large (ω/ω_n) and ϕ equal to π. The dotted line is the observed behavior when $\zeta = 0$, which corresponds to no damping. Figure by MIT OCW.