2.003J/1.053J Dynamics and Control I, Spring 2007
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3/12/2007

Lecture 10

2D Motion of Rigid Bodies: Falling Stick Example, Work-Energy Principle

Example: Falling Stick

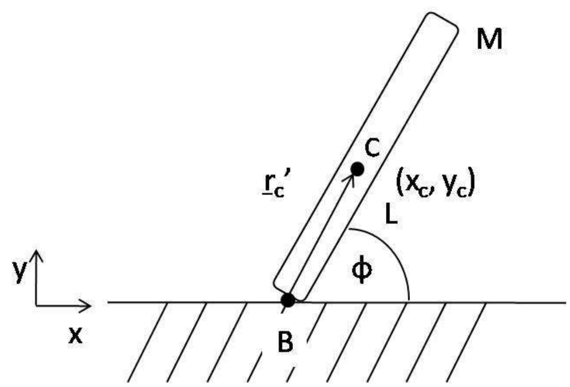

Figure 1: Falling Stick. Stick has mass, M and Length, L. Figure by MIT OCW.

What is the equation of motion if the stick slides without friction along flat a surface?
Application: Joint slipping and sliding.
Uniform Rod: Center of Mass C.

Kinematics

Constraint: $y_B = 0 \Rightarrow$ 3 - 1 generalized coordinates
So we choose x_C, ϕ to be the generalized coordinates because body is rigid. x_B is related to x_C.
Forces not asked for in problem.

$$\underline{r}_C = x_C \hat{e}_x + \frac{L}{2}\sin\phi \hat{e}_y$$

$$\dot{\underline{r}}_C = \dot{x}_C \hat{e}_x + \frac{L}{2}\dot{\phi}\cos\phi \hat{e}_y$$

$$\ddot{\underline{r}}_C = \ddot{x}_C \hat{e}_x + \frac{L}{2}\ddot{\phi}\cos\phi - \frac{L}{2}\dot{\phi}^2 \sin\phi \hat{e}_y$$

Kinetics

Free Body Diagram

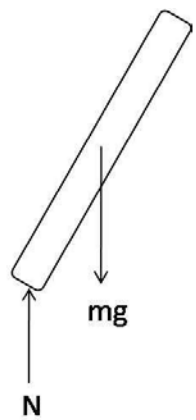

Figure 2: Free Body Diagram of Falling Stick. Figure by MIT OCW.

Which principles to apply?
Want to avoid using N
(1) Angular momentum about B
(2) Linear momentum in x (No forces in x-direction)

Could use work-energy principle, because N does no work.

Linear Momentum in x-direction

$$\underline{F}_x = \frac{d}{dt}\underline{P}_x$$

$$\boxed{0 = m\ddot{x}_c}$$

Angular Momentum about B

$$\underline{\tau}_B = \frac{d}{dt}\underline{H}_B + \underline{v}_B \times \underline{P}$$

$$\underline{v}_B = \frac{d}{dt}x_B\hat{e}_x = \frac{d}{dt}(x_C - \frac{L}{2}\cos\phi)\hat{e}_x$$

$$\boxed{\underline{v}_B = \left[\dot{x}_C + \frac{L}{2}\dot{\phi}\sin\phi\right]\hat{e}_x}$$

$$\begin{aligned}
\underline{H}_B &= \underline{H}_C + \underline{r}_C^{'} \times \underline{P} \\
&= I_C\dot{\phi}\hat{e}_z + \left(\frac{L}{2}\cos\phi\hat{e}_x + \frac{L}{2}\sin\phi\hat{e}_y\right)\times m\left(\dot{x}_c\hat{e}_x + \frac{L}{2}\dot{\phi}\cos\phi\hat{e}_y\right) \\
&= I_C\dot{\phi}\hat{e}_z + m\frac{L^2}{4}\cos^2\phi\dot{\phi}\hat{e}_z - \frac{mL}{2}\dot{x}_c\sin\phi\hat{e}_z \\
&= \left[\left(I_C + m\frac{L^2}{4}\cos^2\phi\right)\dot{\phi} - \frac{mL}{2}\dot{x}_c\sin\phi\right]\hat{e}_z \\
&\neq \left(I_C + \frac{mL^2}{4}\cos^2\phi\right)\dot{\phi}\hat{e}_z
\end{aligned}$$

Note: We cannot write in this problem that $I_B\underline{\omega} = \underline{H}_B$. Point B is moving, not stationary.

$$\boxed{\dot{\underline{H}}_B = \left[\left(I_C + \frac{mL^2}{4}\cos^2\phi\right)\ddot{\phi} - \frac{mL^2}{4}\cdot 2\cos\phi\sin\phi\dot{\phi}^2 - \frac{mL}{2}\ddot{x}_C\sin\phi - \frac{mL}{2}\dot{x}_C\dot{\phi}\cos\phi\right]\hat{e}_z}$$

$$\underline{v}_B \times \underline{P} = \left(\dot{x}_C + \frac{L}{2}\dot{\phi}\sin\phi\right)\hat{e}_x \times m\left(\dot{x}_C\hat{e}_x + \frac{L}{2}\dot{\phi}\cos\phi\hat{e}_y\right)$$

$$\boxed{\underline{v}_B \times \underline{P} = \left(\dot{x}_c m\frac{L}{2}\dot{\phi}\cos\phi + \frac{mL^2}{4}\sin\phi\cos\phi\dot{\phi}^2\right)\hat{e}_z}$$

$$\underline{\tau}_B = -mg\frac{L}{2}\cos\phi\hat{e}_z$$

Notice all terms are in the $\hat{e}_z$ or $-\hat{e}_z$ direction.

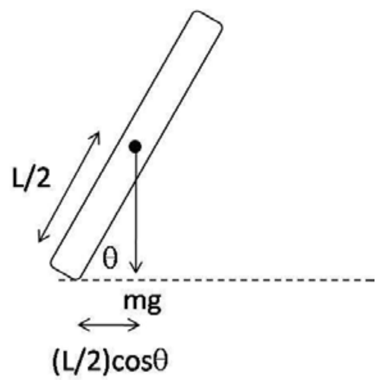

Figure 3: Free Body Diagram of Falling Stick. Figure by MIT OCW.

Therefore:

$$
\begin{aligned}
-mg\frac{L}{2}\cos\phi &= \left(I_C + \frac{mL^2}{4}\cos^2\phi\right)\ddot{\phi} - \frac{mL^2}{2}\cos\phi\sin\phi\dot{\phi}^2 - \frac{mL}{2}\dot{x}_C\dot{\phi}\cos\phi \\
&+ \frac{mL}{2}\dot{x}_C\dot{\phi}\cos\phi + \frac{mL^2}{4}\sin\phi\cos\phi\dot{\phi}^2 \\
&= \left(I_C + \frac{mL^2}{4}\cos^2\phi\right)\ddot{\phi} - \frac{mL^2}{4}\cos\phi\sin\phi\dot{\phi}^2
\end{aligned}
$$

$$
\boxed{-mg\frac{L}{2}\cos\phi = \left(I_C + \frac{mL^2}{4}\cos^2\phi\right)\ddot{\phi} - \frac{mL^2}{4}\cos\phi\sin\phi\dot{\phi}^2}
$$

$$
\boxed{0 = m\ddot{x}_C}
$$

Alternative Approach: Apply Angular Momentum About Point C, The Center of Mass

We should get the same answer by applying angular momentum about C.

$$
\underline{\tau}_C = \frac{d}{dt}\underline{H}_C = \frac{d}{dt}I\underline{\omega} = I_C\ddot{\phi}\hat{e}_z
$$

$$
-N\frac{L}{2}\cos\phi\hat{e}_z = I_C\ddot{\phi}\hat{e}_z
$$

$$
\boxed{I_C\ddot{\phi} = -N\frac{L}{2}\cos\phi}
$$

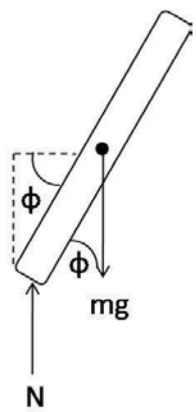

Figure 4: Free Body Diagram of Falling Stick. The approach employed here is one that uses angular momentum about C. Figure by MIT OCW.

Simpler expression but must find N. Use linear momentum in y-direction.

$$N - mg = m\left(\frac{L}{2}\ddot{\phi}\cos\phi - \frac{L}{2}\dot{\phi}^2\sin\phi\right)$$

From above:

$$N = \frac{-2I_C\ddot{\phi}}{L\cos\phi}$$

$$2I_C\ddot{\phi} + mgL\cos\phi = -ml\cos\phi\left(\frac{L}{2}\ddot{\phi}\cos\phi - \frac{L}{2}\dot{\phi}^2\sin\phi\right)$$

$$\boxed{-mg\frac{L}{2}\cos\phi = \left(I_C + \frac{mL^2}{4}\cos^2\phi\right)\ddot{\phi} - \frac{mL^2}{4}\cos\phi\dot{\phi}^2}$$

Notice the equations are the same.

Work-Energy Principle for Rigid Bodies

We need a kinetic energy expression.

If you can show that all non-conservative external forces do no work, $V + T =$ Constant. V is potential energy defined based on center of mass.

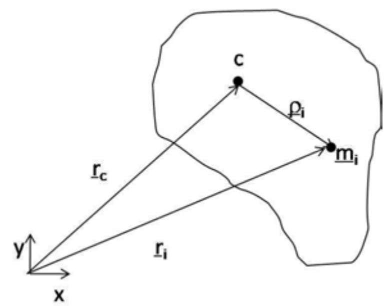

Figure 5: Rigid Body. Figure by MIT OCW.

$$\begin{aligned} T &= \frac{1}{2}\sum_i m_i \underline{v}_i \cdot \underline{v}_i \\ &= \frac{1}{2}\sum_i m_i (\underline{v}_C + \underline{\omega} \times \underline{\rho}_i) \cdot (\underline{v}_C + \omega \times \underline{\rho}_i) \\ &= \frac{1}{2}\sum_i m_i \Big[\underline{v}_C \cdot \underline{v}_C + 2\underline{v}_C \cdot (\underline{\omega} \times \underline{\rho}_i) + (\underline{\omega} \times \underline{\rho}_i) \cdot (\underline{\omega} \times \underline{\rho}_i)\Big] \\ &= \frac{1}{2} M v_c^2 + \underline{v}_C \cdot \underline{\omega} \times \sum_i m_i \underline{\rho}_i + \frac{1}{2}\sum_i m_i \underline{\rho}_i^2 \underline{\omega} \cdot \underline{\omega} \end{aligned}$$

$\sum_i m_i \underline{\rho}_i = 0$ because of center of mass.

$$\boxed{T = \frac{1}{2} M v_C^2 + \frac{1}{2} I_C \omega^2}$$