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2.002 MECHANICS AND MATERIALS II
HOMEWORK NO. 2

Distributed: Wednesday, February 18, 2004
       Due: Wednesday, February 26, 2004

Problem 1 (15 points)

The <u>stiffness</u> of a linear elastic structure subjected to a force of magnitude "F", which results in a displacement of the load point in the direction of the force of magnitude "δ", is given by
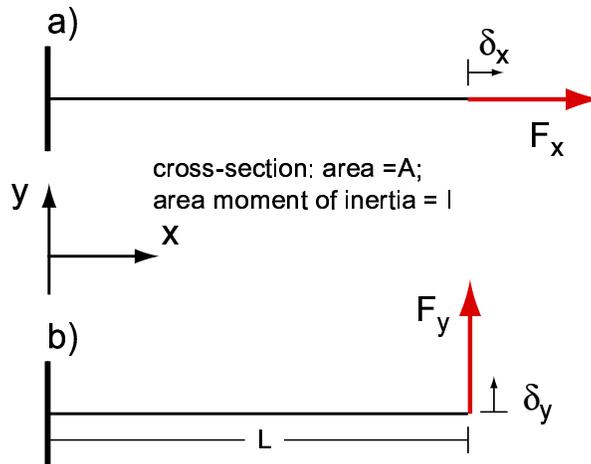
$$k \equiv \frac{F}{\delta}.$$

Figure 1: Cantilever beam: a) axial loading; b) lateral loading.

Consider the case when the structure is, say, a tip-loaded cantilever beam, as shown in the figure. Depending on the orientation of the applied load with respect to the beam axis, two different stiffness values result. When loading is <u>parallel</u> to the beam axis, the **axial stiffness** of the structure can be defined as

$$k_{\text{axial}} \equiv \frac{F_x}{\delta_x},$$

where it is understood that δ_x is the axial displacement of the tip under load F_x. When the applied load is <u>perpendicular</u> to the axis of the cantilever, a **bending stiffness** can be defined as

$$k_{\text{bending}} \equiv \frac{F_y}{\delta_y},$$

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where δ_y is the lateral displacement of the tip under transverse load F_y.

The beam is of length L, and has cross-sectional area A and area moment of inertia I. Under the usual assumptions of slender beam-bending theory, L is much greater than linear dimensions of the beam cross-section, including both $\ell_1 \equiv \sqrt{A}$ and $\ell_2 \equiv \sqrt{I/A}$; that is, $L \gg \ell_1$ and $L \gg \ell_2$.

Show that the ratio of a slender cantilever's bending stiffness to its axial stiffness is extremely small, and give a precise expression for this ratio in the case of a solid circular cross-section of diameter $d \ll L$.

Problem 2. (45 points)

When a cantilever beam is subjected to a temperature change that varies <u>linearly</u> through its thickness (but does not vary along the length of the beam), the beam <u>deflects</u> laterally with a constant curvature, $\kappa_{(\text{thermal})}$, with resulting lateral displacement field

$$v(x) = \frac{1}{2}\kappa_{(\text{thermal})}x^2.$$

Similarly, the mid-surface axial displacement field is given by

$$u_0(x) = \epsilon_{0(\text{thermal})}x,$$

where $\epsilon_{0(\text{thermal})}$ is a dimensionless constant. Here it is understood that the base of the cantilever is at $x = 0$, while its tip is at $x = L$.

Let the beam cross-section be rectangular, of width b (in z-direction) and thickness h (in y-direction), and let the temperature change vary linearly with the y-coordinate according to

$$\Delta T(y) = \Delta T_0 - \frac{\Delta T^\star y}{h/2},$$

so that the temperature change at mid-thickness ($y = 0$) is ΔT_0, at the bottom of the beam ($y = -h/2$) it is $\Delta T_0 + \Delta T^\star$, and at the top of the beam ($y = +h/2$) it is $\Delta T_0 - \Delta T^\star$. (Aside: according to Fourier's law of heat conduction, linearly-varying temperature fields such as this are associated with a steady flow of heat from the higher-temperature region to the lower one.)

- Show that the curvature of the thermally-loaded cantilever is given by

$$\kappa_{(\text{thermal})} = \frac{2\alpha\Delta T^\star}{h},$$

  independent of the value of ΔT_0, and that the axial mid-surface strain is given by

$$\epsilon_{0(\text{thermal})} = \alpha\Delta T_0,$$

  independent of the value of $\Delta T^\star$.

- Show that there is **zero axial stress everywhere** in the thermally-loaded cantilever just described.

If, instead, the thermally-loaded structure is also subjected to a boundary condition at its tip that requires zero lateral displacement ($v(x = L) = 0$), then non-zero reaction forces and axial stress will be generated.

- Using [thermo-]elastic superposition, evaluate the magnitude of the tip reaction force, R_{tip}, required to bring the total tip deflection in the thermally-loaded and tip-restrained cantilever back to zero; i.e.,

$$0 = v_{\text{total}}(x = L) = v_{\Delta T}(x = L) + v_{R_{\text{tip}}}(x = L),$$
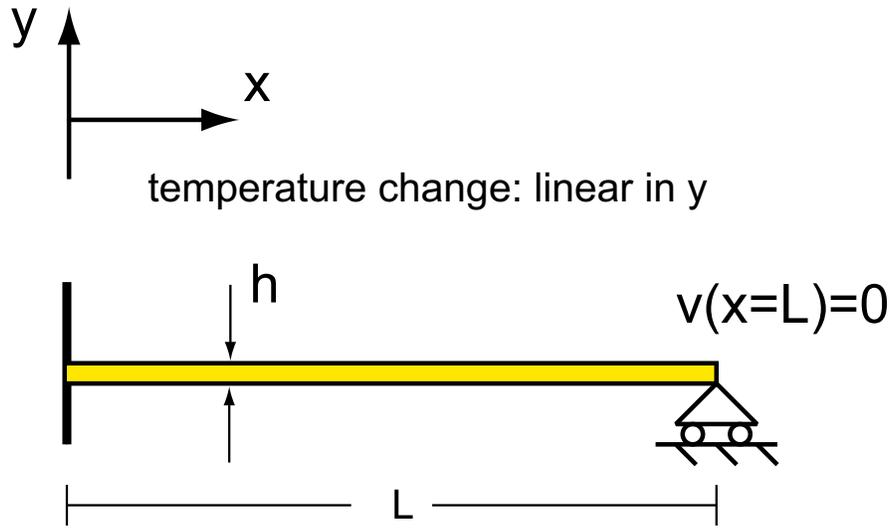
Figure 2: Tip-restrained cantilever beam subjected to a temperature change varying linearly with through-thickness coordinate, y: $\Delta T(y) = \Delta T_0 - (\Delta T^\star y)/(h/2)$.

where $v_{\Delta T}(x = L)$ is the tip displacement of the free thermally-loaded cantilever (above), and $v_{R_{\mathrm{tip}}}(x = L)$ is the tip displacement of the cantilever in response to lateral load R_{tip} applied at $x = L$.

- Identify the location of maximum tensile stress in the thermally-loaded cantilever having restrained tip displacement $v_{\mathrm{total}}(x = L) = 0$, and symbolically evaluate the tensile axial stress at that location.

- Suppose that the beam with tip restraint is made of steel, with $E = 210\,GPa$, $\alpha = 12 \times 10^{-6}/^\circ K$, and has dimensions $L = 100\,mm$, $b = 15\,mm$, and $h = 3\,mm$. What is the largest bottom/top difference in temperature change, parametrized by $\Delta T^\star_{\mathrm{max}}$, so that the steel does not reach a yield strength of $\sigma_y = 350\,MPa$?

Problem 3. (20 points)

The mass density of a crystal can be calculated from a knowledge of the unit cell of the crystal, the atoms within the cell, and their atomic weights. As an example, consider iron (FE), which has a body-centered cubic (BCC) crystal structure. A unit cell contains nine (9) iron atoms, with 8 of them located on the corners of a cube of edge length $r_0 = .2866\,nm$, plus one additional iron atom at the center of the cube. By translations of this unit cell in the three principal cube directions, an arbitrarily large crystal can be generated. Evidently, each corner atom of the cell "participates" in eight $(8 = 2^3)$ different unit cells of this sort; thus only 1/8 of the mass of the corner atoms should be assigned to "this" unit cell. Reflection will show that the net number of iron atoms in the cell is $2 = 1 + 8 \times (1/8)$, where "1" denotes the central atom.

Mass density is defined as

$$\rho \equiv \frac{\text{mass}}{\text{volume}};$$

when applied to a unit atomic cell, we can generalize this result as

$$\rho = \frac{\sum_i (\text{atom/cell}) \times (\text{mass/atom})}{\text{volume/cell}}.$$

The atomic mass of iron (check out a periodic table) is 55.847 g/mol, and the Avogadro number is $N_A = 6.02 \times 10^{23}$ atoms/mol, so the atomic mass of iron is $(55.847/6.02) \times 10^{-23}$ g/atom. Show that the calculation outlined predicts a lattice-based estimate of the density of iron as

$$\rho_{\text{lattice}} = 7.88 Mg/m^3,$$

which is slightly larger than the measured density of iron, $7.87\,Mg/m^3$. The slight discrepancy is due to the presence of defects in the actual crystal; defects were ignored in the model.

Sodium chloride (salt!) is an ionic crystal for which a simple unit cell can be constructed, consisting of alternating sodium (Na) and chlorine (Cl) atoms spaced a distance r_0 apart along lines of a cubic grid. An 8-atom section of such a cell is shown. Assuming the density of salt is $\rho_{NaCl} = 2.17\,Mg/m^3$, estimate the lattice spacing, r_0.
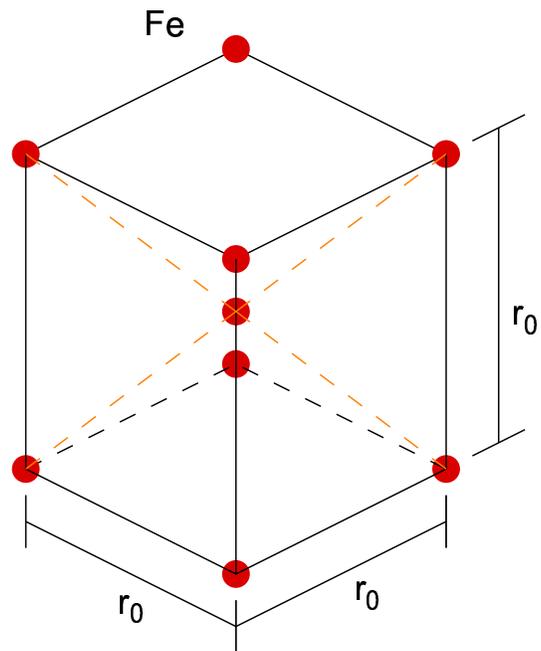
Figure 3: Unit atomic cell (BCC) for Fe. Dashed orange lines connect central atom to corner atoms. (Not all shown, for clarity.)
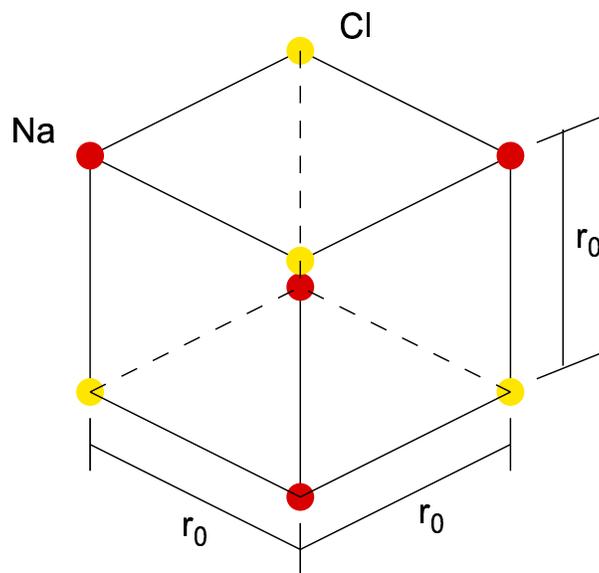

Figure 4: Unit atomic cell for NaCl.

Problem 4. (20 points)

In our simple model of atomic spacing (r_0) and atomic bond stiffness (S_0), we argued that the elastic modulus, E, should scale as

$$E \doteq \frac{S_0}{r_0},$$

and concluded that the orders of magnitude differences observed in E among solids, along with only a factor of two or so variation in r_0, pointed clearly to the dominant role of bond stiffness, S_0, in scaling elastic modulus. We further grouped dominant classes of [primary] bonds as covalent, metallic, ionic, and weak polarized bonding (van der Waals; hydrogen bonding).

Although the extreme simplicity of the underlying model precludes much more precise evaluation, we can push things a bit further. Several metalic elements (nickel (Ni), gold (Au), copper (Cu), silver (Ag), aluminum (Al), palladium (Pd), platinum (Pt), and others) form a face-centered cubic (FCC) lattice, and are, of course metallic. Within a single crystal class (here, FCC), our model might lead us to expect that the inferred strength of the [metallic] bonding, S_0, should not vary too much over the class. The table lists measured ambient-temperature elastic moduli (E) and FCC lattice spacings (here denoted "a_0" rather than "r_0") for some of the FCC metals:

Metal	E (GPa)	a_0 (Å)
Ni	214	3.1517
Al	70	4.0496
Pt	172	3.9231
Pd	124	3.8902
Cu	124	3.6151
Au	82	4.0786
Ag	76	4.0862

- Evaluate an inferred value for the bond stiffness parameter, S_0, for each FCC material.

- Do you observe any patterns in your results? For example, do particular pairs (or so) of metals give very similar results for S_0? Although the precise nature of the connections resides in deeper within quantum mechanics, suffice it to say that location in the periodic table might have something to do with the answer...