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PROFESSOR:

So let's start with a simple but quite illustrative example. So suppose you're a bookie. And what a bookie does-- he sets bets on the horses, sets the odds, and then pays money back. Probably collects a fee somewhere in between.

So suppose he is a good bookie and he knows quite well the horses, and there are two horses. He knows that for sure one horse has 20% chance of winning and another horse has 80% chance of winning. Obviously, the general public doesn't have all of this information. So they place a bet slightly differently. And then there is \$10,000 bet on one horse and \$50,000 bet on another horse.

Well, bookie is sure that he possesses good information. So he-- suppose he sets the odds according to real life probability. So he sets it four to one. What would be possible outcomes of the race for him? Monetary.

So suppose the first horse wins. Then what happens? He has to pay back \$10,000 and four times more. So he pays out \$50,000. And he receives \$60,000, right? So he can keep \$10,000 out of it. OK.

So what happens is the other more probable horse wins. Well, he'll have to pay back the \$50,000 and one quarter of it, which is \$12.50. So at the end, he'll pay 62 1/2 thousand, while he collected \$60,000, out right?

So he will-- in this situation, he will lose \$2,500. Well, all in all, he expects to make nothing. So he probably could collect enough fees to cover his potential loss. But there is certainly a variability in outcomes. He can win a lot. He can lose some.

Now, what if he forgets about his knowledge about the real life probabilities of horses winning or losing and instead sets bets according to the amount which we

are already bet. According to the market, effectively. So what if he sets the odds five to one, according to the bets placed?

Well, in this situation, if the first horse wins, he pays back 10 plus 5 times 10, so 60. He is 0. And if the second horse wins, he pays back 50 plus 1/5 of 50, plus another 10. Again 60.

So no matter which horse wins, he will get 0. We're 100% sure. And if he collects some fee on top of it, he will make a riskless profit. And that's how, actually, bookies are operating.

So it's a simple example. But it gives us a first idea of how a risk-neutral framework and risk-neutral pricing works. So we are, here, not in the business of making bets on horses. We are actually in the business of pricing derivatives. So we will talk about the simplest possible derivatives-- mostly derivatives on stocks. But there are more complicated derivatives which underline for which could be interest rates, bonds, swaps, commodities, whatever.

So a derivative contract is some-- in general speaking, a formal pay-out connected to underline. Usually, the underline is a liquid instrument which is traded on exchanges. And derivative may be traded on exchanges. Actually, quite a few equity options are traded on exchanges. But in general, they are over-the-counter contracts where two counter parties just agree on some kind of pay-out.

One of the simpler derivatives is a forward contract. So what is a forward contract? A forward contract is a contract where one party agrees to buy an asset from another party for a price which is agreed today. Usually, this forward price is set in such a way that right now, no money changes hands. Right?

And here is an example. Well, suppose there is a stock which, right now, is priced at \$80. And this is the forward for two years. So somebody agrees to buy the stock in two years for this price. And not surprisingly, I somehow set this price such that currently the value of the contract is 0.

And we'll see how I'll come up with the price. So this blue line is actually the pay-out

what will happen at the end. Right? The pay-out, depending-- the graph of F at time T to a determination time or expiry-- how it depends on the stock price. Right?

So obviously, the pay-out is S minus K , where S is the stock price, so it's a linear function. It turns out that the counter price is also a linear function but slightly shifted. And we'll see how come it's slightly shifted and how much it should be shifted. And K is usually referred to as a strike price.

Another slightly more complicated contract is called a Call option. So if previously the forward is an obligation to buy the asset for an agreed price, Call option is actually an option to buy an asset at the agreed price today. You can view it-- a Call option can be viewed as kind of insurance that the-- against the asset going down.

Basically the pay-out is always positive. You can never lose money. On the forward, you can lose money. You agree on the price. The asset ends up being lower than this price, but you still have to buy it. Right?

Here, if the asset ends up at expiry below strike price or out of the money, then the pay-out will be 0. If, on the other hand, it ends up being above the strike price or it's called, the option is in the money. Then the pay-out will be S minus K as before.

So in mathematical terms, the pay-out is maximum of S minus K and 0. Right? And that's what happens at expiry time-- this blue line. So what is the price of this option now? Well, obviously it should be slightly above because even if now the asset is slightly out of the money-- below strike price-- there is some volatility to it, and there is a probability that we will still end up in the money at expiry. So you would be willing-- you should be willing to pay something for this.

Obviously, if it's way out of the money, it should be 0. Right? On the other hand, if it's way in the money, in fact, it should be just as forward. And in fact, it is. We'll see because the probability for the asset going back to the strike price and below will be low. And the Black-Scholes equation and Black-Scholes formula is exactly the solution for this curved line, which we'll see in a second.

Another simple contract, which is kind of dual to Call option, is a Put option. So Put

option, on the contrary, is a bet on the asset going down, rather than up. Right? So the pay-out is maximum of $K - S$ and 0. So it's kind of reversed.

Also a ramp function, at maturity. And here is the current price. Again, even if it's in the money-- if it's way in the money, we expect it to be 0. If it's way in the money, we expect it to be slightly below forward, just because of this counting.

OK. So and here are a few-- three main points, which we'll try to follow, through the class. So first of all, what we'll see-- that if we have current price of the underline and some assumptions on how the market or the underline behaves, there is actually no uncertainty in the price of the option, obviously, if we fix the pay-out. Right?

So somehow there is no uncertainty. It's completely deterministic, once we know the price of underline. The other interesting fact, which we'll find out, is actually risk-neutrality, meaning that in fact, the price of the option has nothing to do with the risk preferences of market participants or counter-parties. It actually only depends on the dynamics of the stock, only depends on the volatility of the stock. And finally, the most important idea of this class-- that mathematical apparatus allows you to figure out how much this deterministic option price is now.

So let's consider a very simple example, a very simple market, two-period. So suppose our time is discrete, and we are one step before the maturity. So right now, our stock has price at 0. And there is some derivative of 0 with some pay-out. We'll consider a few of those. Right?

Also, we'll add to the mix a bit of cash. Right? Some amount of riskless cash would be 0. And riskless meaning that it grows exponentially with some interest rate R . And there is no uncertainty. It's completely-- if you have now the 0, we know then, in time dt , our B_0 will grow exponentially. It will become B equals RT . So at 1, basically, 0 [INAUDIBLE]-- money market account, rather. If you go to Cambridge Savings Bank, put \$1 in today, then in a year, you'll get \$1 and basically nothing because interest rates are 0.

So in time, dt , we will assume with some probability, P , our market can go to the state where stock becomes S_1 -- the price of stock becomes S_1 . Our bond grows exponentially-- no uncertainty. And our derivative becomes f_1 . Or with probability 1 minus P , only two states. So our stock becomes S_2 . Bond stays the same. And the derivative is some f_2 .

So let's start with our simple contract, the Forward contract. So one can naively approach a problem, trying to get the price of the derivative, using the real world probabilities, P and 1 minus P . Right?

So we know that the pay-out is S minus K . That's given. So one would say that if we know we are one step before the pay-out, so let's just compute expected value of the pay-out, using real world probabilities, get this value. And actually, what we are looking here is to set K such that the price now at time T is 0 . That's usual convention.

So we'll then set K to this probability, to this number, which depends on real world probability and obviously depends on the stock price at expiry. But obviously, we don't know real world probabilities. We can guess. We can say, oh, this stock is as likely to go up then down. Then it's just an average of a end stock prices or something else. But it's all [INAUDIBLE] wavy. And actually, we never will be right.

Instead of doing this-- we're kind of following bookie example-- let's try to do something else. Let's think a little bit. So we have a stock which is trading at market now for the price of 0 . How about we go to the bank and borrow $\$S_0$ right now and immediately go to the market and buy the stock.

So right now we are not 0 . We borrowed at 0 . We paid it immediately to buy the stock. So we have stock at hand. Then we'll wait for one period. And at the same time-- sorry-- we enter on the short side of the Forward contract. So we agree to sell the stock for some price, K_0 .

So in dt , in a month period of time, the contract expires. We already have stock. So we just go and exchange it for $\$K_0$. Right? But at the same time, we need to repay

our loan which now have become S_0 times e to the rdt . This is deterministic, right? We bought it at 0. In time, dt , it became S times e to the rdt .

So what's our net? The net is K_0 minus S times e rdt . So suppose K_0 is greater than this value. Then we made riskless profit. There is no risk in the strategy which we proposed. So this is good. But why wouldn't everybody do it all day long?

On the other hand, if K_0 is less than S_0 , that's a loss for sure. And if anybody thinks, as we did-- and we assume that everybody can do it-- then nobody you would want to enter it, which means that in order for our forward to be price 0 now, the strike price has to be equal to this amount. And there is no uncertainty about it.

So let's stop and think a little bit. Well, actually, just to see how it works. And that's exactly why I set this K to this number. So by the way, who can tell me which interest rate does it imply? If our strike-- our stock price is \$80, our strike is 88.41. And the expiry is in two years, approximately.

AUDIENCE: 2.5?

PROFESSOR: 2.5. So in two years, it will be 5%. So roughly speaking, without compounding, it should be 5% of 80 plus 5%. It would be 84. So 10% for two years. So the interest rate is 5%. Yeah. So yeah. That's actually exactly 5 with exponential component.

Yeah. Well, in a good world-- probably five years ago, that's how it would work. The two years interest rates now, the last time I checked, was, I think, 30 [INAUDIBLE]. We can check where the bond is trading now.

All right. Give me a sec. Now. Yep. 32 1/2 basis points. 1.6 basis points up, since the morning. Quite a bit, by the way. So yeah. So right now interest rates are basically 0. So these two lines would be very close right now if we were for two years, in that case.

So coming back to our example. So what's important here? How did we arrive to this strike price, or to this price of the Forward contract? We, in fact, tried-- we took some amount of stock. In this particular case, it was the whole price of stock. We

took some amount of cash, and by combining these two pieces, we somehow replicated the final pay-off. Right?

And that's the general idea of risk neutral pricing and replicating portfolio. What we will try to do, in the rest of the class, is take a pay-off and try to find a replicating portfolio, maybe more complicated, maybe a dynamic such that at the end, this replicating portfolio will be exactly our pay-off. Right?

And what would it mean? Well, obviously it would mean that the current price of the derivative should be the price of our replicating portfolio right now. Right? And that's how the risk neutral pricing works.

So we are still in this simple situation. But we will try to price a general pay-off, F_1 -- a general pay-off, F . Right? And here's how it goes. So we still will try to form our replicating portfolio out of the bond of some amount of bond and some amount of stock.

And we'll say that we will need a , S_1 , and b of the bond. Right? And we'll try to find a and B such that no matter what the real world probability is, at one step maturity, we'll replicate our pay-off exactly.

And fortunately, in this particular case, it's very doable. It's just two equations. We use two variables. We should be able to do it. And we can solve it and find this a and b .

Then we'll substitute them in the formula. Right? Take the current price of the stock, which we know, and some cash, and find the current price of the derivative. Right?

And this works-- it should work for any derivative. It doesn't matter is it Forward, Call, Put, or some complicated option, as long as it is deterministic at expiry. An interesting way, though, to look at it is to rewrite this formula slightly, in such a way, which does remind us, taking an expected value, maybe discounting it because this is expected value, at some time in the future.

But this probability-- and it is a probability because this number q , here, is between

0 and 1. But this probability has little to do with real world. Right? In fact, it's something different.

But that probability exists. And it's called-- the measure where our stock behaves like this is called a risk neutral measure or martingale measure. And in this measure, as we will see, the value of the derivative will be just expected value of our pay-out. And that's-- yeah. That's what I'm trying to say, here.

So now let's get into continuous world. Right? In continuous world, we'll need some assumptions on the dynamics of our stock underline. And let's make an assumption that it is log normal. What does it mean that it's log normal? It means that the proportional change of the stock, over infinitely small amount of time, dt , has some drift, μ , and some stochastic component, which is just Brownian Motion. Right?

So this dW is distributed normally with mean 0 and standard deviation, which is actually square root of dt . That's how Brownian Motion works. And that's extremely important, that the standard deviation of Brownian Motion is square root of Δt . And that's how it works.

And again, we will use this idea of replicating portfolio. What would it mean in this case? Well, we would like to find such $[\Delta A, \Delta B]$ a and b, on this infinitely small period of time, dt , such that by combining small changes in stock, with $[\Delta A, \Delta B]$ A, and small changes in bond, with $[\Delta A, \Delta B]$ B, will exactly replicate the change in the derivative-- in the pay-out of derivative-- not pay-out. In the derivative. In the change of the derivative, over this infinitely small time, t .

Well, to do this, we'll need to use Ito's formula. Did you talk about Ito already? OK. Cool. That's great.

So just to remind you that Ito's formula is nothing more than the Taylor rule, actually-- the first approximation up to dt . But because of the standard deviation of the Brownian Motion being on the scale of square root of t , we will need one more term there. Right?

So one term is d^2S/dt^2 by dt . Another is d^2S/dS^2 by dS . And the square of dS now is

actually of order of magnitude of dt . So we'll need a quadratic term there. All right.

So if this is our df , so what we'll do-- we'll differentiate. We'll just substitute it here. Right?

We'll substitute it here. We'll substitute df taken from our dS , which is like this ndB . Let's not forget that dB -- that B is deterministic. Right? There is nothing uncertain about it. So dB is actually $rBdt$. All right? Because our B grows exponentially with interest rate, r .

So we substitute everything into the formula above. This is just our df with dS expanded and everything. And then when we start comparing the terms. One immediate thing to notice-- that a has to be equal to df or dS , for this to hold. Right?

And if you compare the terms, dt , we'll get this expression here. But that's actually even more the most important part. Then we'll go and use our knowledge that some part of our equation is deterministic and basically take f and aS on one side and leave the deterministic part, on the other side, differentiated once again. And left side will be just $[? rBdt. ?]$ And if we substitute once again df . And don't forget that what we learned is that a is equal to df by dS .

Then we collect all of the terms and arrive to this partial differential equation which connects-- which basically is a partial differential equation for the current price of a derivative-- of any derivative. And how if we solve it, then we should actually be able to know the price of the derivative.

So now how do we solve this partial differential equation? Well, for-- yeah. So a few observations about this equation. Well, the first observation is that any tradable derivative-- we made no assumptions about the pay-off. So any tradable derivative as any pay-off should satisfy this equation.

The other observation is as we expected, there is no dependency on real world drift or any probability of it going up or down. The only dependence is on the volatility of the stock. Right? Not only we found the value of the derivative-- most importantly, we actually were able to come up with a hedging strategy.

And what does it mean, we came up with a hedging strategy? Well, we found coefficients-- for any time, we found the coefficients, a and b , such that we have a replicating portfolio. So what we could do, at any point of time, we can hold the short derivative and long the portfolio of stock, itself, and some cash, and then know how much it should be.

Here, it's more complicated. We have to dynamically change these numbers, as time develops. Every time, dt will have to rebalance. But both parts will replicate each other perfectly.

It's like in a bookie's example. We can go to a counterparty, agree for some derivative contract. For all, there will be some fee. And then we'll go to exchange and buy the stock, and we will get just cash from the bank. And we'll maintain this at some amount of stock and some amount of cash.

And we'll be sure that we are hedged. There is no risk in this combination of the derivative and our hedge. So we will just collect a fee on the transaction.

So that's what actually-- how the business is working. Traders are trading and hedging their positions immediately. I mean, they do take some market risks. But you want to take very little and very directional, very specific market risks and not everything.

So our strategy allows us to have a hedging portfolio at the same time-- hedging strategy. And now there are more mathematical but practical consequences that actually, by certain-- not very easy-- change of variables, we can take the Black-Scholes equation and put it back to heat equation.

Actually, I suggest, as one of the topics for the final paper, for you to do it or check it out in the books. Go and understand it. But the good part of it-- that heat equation is well known and well understood. There are many, many ways to solve it numerically for simple pay-outs for Calls and Puts. We don't have to do it numerically, but if the pay-outs are more complicated or the dynamics is different, then numerical

methods will be needed, for sure.

So again, to solve this equation, we'll need, as for any partial differential equation, we'll need some boundary and initial conditions. And these come from our final pay-out of the option, which we know. We will know what happens at expiry.

And some boundary conditions for Call and Put, the final pay-out is we know. Right? So at time, t . And the boundary conditions we discussed, we can observe them graphically.

So basically for Call, as we said, at current time, t , and boundary 0, it should be 0. The price should be 0. And at infinity, it should be actually the Forward price. So it should be just discounted S minus K . Discounted pay-out. Right? And similarly for Put.

So given these conditions, we can solve the equation. And as I said, for Call and Put and for simple dynamics-- Black-Scholes dynamical or log normal dynamics-- actually, these equations can be solved exactly-- exactly meaning up to this term, the normal distribution, which still has to be computed numerically, obviously. But here are the formulas.

They do kind of look a little bit-- and we'll see about it-- there is some kind of expected volume going on. Right? One probability times another. But these are the formulas. And that's how I drew the lines on the graphs.

And as I said, in fact, the whole world, instead of solving the whole partial differential equation, we can approach it from a risk neutral position and say that, in fact, the price of our derivative now is just expected value of pay-out, discounted, probably, from the maturity. But not in real time or real world measure, but in some specific risk neutral measure.

And how do we find this risk neutral measure? Well, the risk neutral measure is such that the drift of our stock is actually interest rate. It's riskless. That's exactly how we saw it in our binary example. All right?

So even in our binary example, our expected value of our stock, under risk neutral measure, meaning using the risk neutral probability, was drifting with interest rate, r . So they're the same. It happens in continuous case.

And that's another good exercise-- and I would accept it as a final paper-- is deriving the Black-Scholes formula just by the expected value of the Call and Put pay-out with the log normal distribution-- terminal distribution. All right.

So for more complicated pay-offs, the life becomes more complicated. And some finite differences should be used for more complicated pay-offs or American pay-offs or past-dependent pay-offs, tree methods or Monte-Carlo simulations. And that's what was happening in real life. Yeah.

Now, since we have, actually, plenty of time, I would like to give an example of how replicating-- idea of replicating portfolio works. I give a couple more examples.

So OK. Here is a Bloomberg screen for foreign options-- Call options on IBM stock. It actually was taken a while ago-- a few years ago.

And so here are different strikes for a Call option. The current price of the stock is \$81.14. And here are the strikes of the Call. So obviously, if the option is way out of the money, meaning the strike is very high compared to the stock price, the value of the option is 0.

If it's way in the money, in fact, it is just S minus K . So S being \$81. And say, the strike being \$55. So it's \$26. Right?

So there is some difference. But actually, here it's a bit small because the difference should be just discounting, as we know. Right? But it's pretty short-dated options. They are probably months long, so there is not much discounting.

So it becomes pretty parallel. It's similar here, right? So I mean, this changes by 5. This changes by 5. It's pretty linear. But it becomes non-linear around the money, around current stock price. Right? So we do observe this behavior.

But to tell you the truth, if you were to-- I didn't put implied volatilities here. But

actually, you would observe that the world is not Black-Scholes, meaning that-- what's the assumption of Black-Scholes. The assumption of Black-Scholes is that every option, for any strike, on a given stock, on a given expiry, would have the same volatility. Right?

So if we went through exercise of implying the volatility according to Black-Scholes formula, from the option price which is traded on the market and the current price, we would find out that, actually, the volatility is not constant with strike. Well, it's actually skewed. Well, actually it is smiled. They would find something like this, which means that Black-Scholes theory is not perfectly good. Right? So something more complicated should be done.

But in some cases, we even don't need to do something more complicated. One example, being so-called put-call parity. Right?

So let's see. Suppose we look at the screen. So we know all prices for all Call options for all strikes. Well, probably will be some granularity, but we know those. But instead of pricing a Call, we need to price a Put.

Somehow, we don't know how the dynamics of our stock looks like. So we have strong suspicion that it's not exactly log normal. So there is some volatility smile. It's not constant. The world is slightly not Black-Scholes.

So how do we price Put? Well, let's see. We'll stare long enough at the pay-outs of the Call and Put. So what's the pay-out of a Call with some strike? It looks like this. Right?

The pay-out of the Put, with the same strike, would look like this. So what if we take, we buy a Call and sell a Put? So this would go like this. Right?

Straight line. Looks very much like Forward, right? So if we actually subtract the stock from here, move it from here, then it should be-- yeah-- minus K . Yeah.

I think I got the signs correct. Right? And this is just a number. Right?

And that's what happens at pay-out. So if we take this portfolio, if we action now, buy a Call, sell a Put, and sell a stock, we know that at the end, we'll for sure get the K in money. Right?

So which means that now-- so this is at time, t . So right now, it looks, to me, that if we do write this, and that's just the current price of the stock, this should be-- right? We just need to discount this price to now, in this amount of cash, which means that our Put, at any time, t , is stock minus K . Right?

So if we know all of the prices for any strike, K -- if we know price of a Call, we don't need any Black-Scholes or anything. We can immediately tell everybody how much is a Put. Right? So then this relationship is actually a call-put parity.

And that's, again-- that's a replicating portfolio. It's a simple replicating portfolio. It's static, meaning that we fixed it now and we don't change it to expiry. So it's quite good this way. But that's how it works.

Another example. So for this, I have, actually, a picture. So again, we have the same situation. We have prices of Calls. But instead of pricing a Call, we want to price a Digital.

So what is Digital? Digital is such a weird contract, which pay-out is just a function. Basically, it's a bet on the stock to finish above strike price, K . Right?

If at expiry, the stock is above K , you get 1. You'd get \$1. If it's below, you'd get nothing-- 0. Right? So

So such an interesting contract. The question is, can we price it, given that we know the prices of Calls? And I suggest we use the idea of replicating portfolio. Any ideas how to do it?

It's my typical interview question. So just pretend that you are interviewing. Yep?

AUDIENCE:

You [INAUDIBLE] the call, and then you short the call, just like smaller or the higher strike.

PROFESSOR: Yep. The call strike. Yeah, you're absolutely right. Good. You've got an offer.

Yeah. So here's how it goes. So this is a strike K . Right? So let's buy a Call with strike $K - 1/2$ and sell a Call with strike $K + 1/2$. Right? We just sold. So if we combine these two-- well, actually, if this is 1-- yeah. If this is 1, it should look something like this. Great.

So how will it look like? So obviously, here, it's 0. Right? Then it will be like this. Right? And after that, it will be what?

AUDIENCE: Constant.

PROFESSOR: It will be constant. Right? And because this is $K - 1/2$ and this is $K + 1/2$, it will be exactly 1. Right? Good. So our pay-out, at the end, will be like this.

So that's good. But there is quite a bit of slope here. So how can we do better than this? Well, if we buy it at $K - 1/4$, and sell it at $K + 1/4$, and just combine those, it will be exactly the same, but the level will be $1/2$. So we need to buy two of those and to sell two of those. Right?

Well, we might as well go $K - \text{Epsilon}$ and $K + \text{Epsilon}$, so it'll be Call price at strike, K , minus Epsilon, minus Call price at K , plus Epsilon, divided by 2Epsilon . Right? These two Epsilon [? Confucian ?] needed rescale it back to 1. Right?

So in fact, if we go small Epsilon, we need a lot of both. Right? And that's how-- that's the approximation of our Digital price. And that's actually how people on the market do price and hedge, most importantly, the Digital contracts, because Call contracts are liquid, and they are traded on exchanges while Digitals are way less liquid.

So somebody would Call again to counterparty, enter into Digital, and hedge it on the exchange. These two Calls is a Call spread. But now tell me, is it surprising that-- I mean, what does it remind you?

Yeah. So it's derivative of the Call price but with respect to strike. Right? Is it surprising?

How did our Call price look like? It's a ramp. Right? If we take a derivative of this, what will we get? Yeah.

AUDIENCE: [INAUDIBLE].

PROFESSOR: Right. So in fact, if we do something even more weird with this, and then I'll take a square or something else, the same will apply. So it's not surprising at all. All right.

So that's basically how the replicate-- this idea of replicating portfolios is extremely powerful. And in fact, that's what happens in real life. In real life, you have some complicated derivative which you need to hedge.

And how to hedge-- you'll find something else which replicates-- to a certain extent, replicates your pay-off. That's what you'll try to do. And this will be your hedge portfolio. Usually, it's dynamic. So you'll have to rebalance. And that's how you basically reduce the risks.