

18.S096 Problem Set 2 Fall 2013
Probability Theory and Stochastic Process

Due date : **10/3/2013**

Collaboration on homework is encouraged, but you should think through the problems yourself before discussing them with other people. **You must write your solution in your own words. Make sure to list all your collaborators.**

Part A

Part A has problems that straightforwardly follow from the definition. Use this part as an opportunity to get used to the concepts and definitions.

Problem A-1. (Exponential distribution) A continuous random variable X is said to have an exponential distribution with parameter λ if its probability density function is given by

$$f_X(x) = \lambda e^{-\lambda x} \quad (x \geq 0).$$

- (a) Compute the moment generating function of X .
- (b) Compute the mean and variance of the exponential distribution using the moment generating function.
- (c) For a fixed real $t > 0$, compute $\mathbf{P}(X > t)$.
- (d) Prove the *memoryless property*:

$$\mathbf{P}(X > s + t | X > s) = \mathbf{P}(X > t) \quad \forall t, s > 0.$$

- (e) Let X_1, X_2, \dots, X_n be i.i.d. exponential random variables with parameter λ . Prove that $\min\{X_1, X_2, \dots, X_n\}$ has exponential distribution.
- (f) When you enter the bank, you find that all three tellers are busy serving other customers, and there are no other customers in queue. Assume that the service times for you and for each of the customers being served are independent identically distributed exponential random variables. What is the probability that you will be the last to leave among the four customers?

Problem A-2. (Poisson distribution) A discrete random variable X is said to have a Poisson distribution with parameter λ if its probability mass function is given by

$$f_X(k) = \frac{\lambda^k e^{-\lambda}}{k!} \quad (k = 0, 1, 2 \dots).$$

- (a) Compute the moment generating function of X .
- (b) Compute the mean and variance of the Poisson distribution using the moment generating function.
- (c) Prove that the sum of independent Poisson random variables has Poisson distribution. (Hint : use the moment generating function)

Problem A-3. Identify Markov processes.

- (a) Simple random walk.
- (b) The process $X_t = |S_t|$, where S_t is a simple random walk.

- (c) $X_0 = 0$ and $X_{t+1} = X_t + (Y_t - \frac{1}{\lambda})$ for $t \geq 0$, where Y_t are i.i.d. random variables with exponential distribution of parameter λ .
- (d) $X_0 = 0$ and $X_{t+1} = X_t + Z_t Z_{t-1} \cdot Z_0$ for $t \geq 0$, where Z_t are i.i.d. random variables with log-normal distribution.
- (e) $X_0 = 0$ and $X_{t+1} = X_t + W_t + W_{t-1} + \dots + W_0$ for $t \geq 0$, where W_t are i.i.d. random variables with normal distribution.

Problem A-4. (a) Construct an example of a finite Markov chain with non-unique stationary distributions.

(b) A Markov chain with transition probabilities p_{ij} is called *doubly stochastic* if each row of the transition matrix sums to one, i.e.,

$$\sum_{i=1}^m p_{ij} = 1, \quad j = 1, 2, \dots, m.$$

Prove that $\pi_j = \frac{1}{m}$ gives the stationary distribution of a doubly stochastic process over a state space S of size m .

Part B

Part B has more elaborate problems. Many of the problems in Part B cover important topics that we did not have enough time to cover in lecture. Thus understanding the content is as important as solving the problem. Try to think through the content of the problem while solving it.

Problem B-1. Let X_1, X_2, \dots be i.i.d. exponential random variables with parameter λ , and define $\tau = \max\{t \geq 0 : X_1 + \dots + X_t \leq 1\}$. Hence τ measures the number of times an event with ‘exponential waiting time’ occurred during an interval of length 1.

- (a) Compute $\mathbf{P}(\tau = 0)$.
- (b) Compute $\mathbf{P}(\tau = n)$ using the distribution of $S_n = X_1 + X_2 + \dots + X_n$, whose cumulative distribution function is

$$F_{S_n}(x) = 1 - \sum_{k=0}^{n-1} \frac{1}{k!} e^{-\lambda x} (\lambda x)^k \quad (x > 0).$$

Conclude that τ has Poisson distribution.

Problem B-2. (a) Verify that the probability density function of the normal distribution

$$\phi(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)} \quad (-\infty < x < \infty)$$

is indeed a probability density function, i.e. $\int_{-\infty}^{\infty} \phi(x) dx = 1$ and $\phi(x) \geq 0$ for all $-\infty < x < \infty$. (Hint : Compute $(\int_{-\infty}^{\infty} \phi(x) dx)^2$ using polar coordinates)

(b) Compute the expectation and variance of log-normal distribution (Hint : it suffices to compute the case when $\mu = 0$. You can use part (a) in order to simplify the computation)

Problem B-3. A family of pdfs or pmfs is called an *exponential family with parameter θ* (θ is a vector) if it can be expressed as

$$f(x|\theta) = h(x)c(\theta)\exp\left(\sum_{i=1}^k w_i(\theta)t_i(x)\right),$$

where $c(\theta) \geq 0$ and $w_1(\theta), \dots, w_k(\theta)$ are real-valued functions of θ , and $h(x) \geq 0$ and $t_1(x), \dots, t_k(x)$ are real-valued functions that do not depend on θ .

Prove that log-normal distribution, Poisson distribution, and exponential distribution are exponential families.

Problem B-4. It is well believed that investing in the stock market over a long period of time reduces the risk. The following example taken from M. Kritzman, "Puzzles of finance", Chapter 3 (See Kritzman and Rich, "Beware of Dogma: the truth about time diversification") argues against this idea.

Consider a theoretical stock whose annual return has log-normal distribution with parameters μ and σ for $\mu = \ln(1.1)$ and $\sigma = \ln(1.2)$. Assume that the return of each year is independent to other years. For this theoretical stock, the fraction of wealth lost with 0.1% chance when invested over T years is 40.5%, 58.43%, 61.48%, and 73.92%, for $T = 1, 5, 10, 20$ respectively. The author then concludes that:

These results reveal that if risk is construed as annualized variability, then time diminishes risk. However, if the magnitude of potential loss defines risk, then it increases with time.

(a) Compute the fraction of wealth lost with 0.1% chance when invested over T years, as a function of T (the function may involve the cumulative distribution function of the normal distribution).

(b) Find the value of T that maximizes the function $h(T)$ computed in (a). Is it true that $h(T)$ is an increasing function of time?

(c) Criticize the argument above, using the computations of steps (a) and (b).

Problem B-5. A person walks along a straight line and, at each time period, takes a step to the right with probability b , and a step to the left with probability $1 - b$. The person starts in one of the positions $1, 2, \dots, m$, but if he reaches position 0 (or position $m + 1$), his step is instantly reflected back to position 1 (or position m , respectively). Equivalently, we may assume that when the person is in positions 1 or m , he will stay in that position with corresponding probability $1 - b$ and b , respectively.

(a) Model this problem as a Markov chain and describe the transition matrix.

(b) Find the stationary distribution.

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