1.3.2 An open problem about spike models

Open Problem 1.3 (Spike Model for cut–SDP [MS15]. As since been solved [MS15]) *Let W denote a symmetric Wigner matrix with i.i.d. entries $W_{ij} \sim \mathcal{N}(0,1)$. Also, given $B \in \mathbb{R}^{n\times n}$ symmetric, define:*

$$Q(B) = \max\left\{\operatorname{Tr}(BX) : X \succeq 0,\ X_{ii} = 1\right\}.$$

Define $q(\xi)$ as

$$q(\xi) = \lim_{n\to\infty} \frac{1}{n}\mathbb{E}Q\left(\frac{\xi}{n}\mathbf{1}\mathbf{1}^T + \frac{1}{\sqrt{n}}W\right).$$

What is the value of ξ_*, defined as

$$\xi_* = \inf\{\xi \geq 0 : q(\xi) > 2\}.$$

It is known that, if $0 \leq \xi \leq 1$, $q(\xi) = 2$ [MS15].

One can show that $\frac{1}{n}Q(B) \leq \lambda_{\max}(B)$. In fact,

$$\max\left\{\operatorname{Tr}(BX) : X \succeq 0,\ X_{ii} = 1\right\} \leq \max\left\{\operatorname{Tr}(BX) : X \succeq 0,\ \operatorname{Tr} X = n\right\}.$$

It is also not difficult to show (hint: take the spectral decomposition of X) that

$$\max\left\{\operatorname{Tr}(BX) : X \succeq 0,\ \sum_{i=1}^n X_{ii} = n\right\} = \lambda_{\max}(B).$$

This means that for $\xi > 1$, $q(\xi) \leq \xi + \frac{1}{\xi}$.

Reference

[MS15] A. Montanari and S. Sen. Semidefinite programs on sparse random graphs. *Available online at arXiv:1504.05910 [cs.DM]*, 2015.