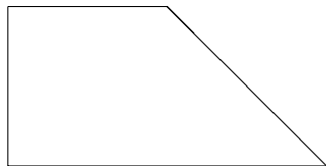


## 18.A34 PROBLEMS #10

111. [1] The following shape consists of a square and half of another square of the same size, divided diagonally.



Cut the shape into four congruent pieces.

112. [1.5] Take five right triangles with legs of length one and two, cut one of them, and put the resulting six pieces together to form a square.
113. [1.5] Find an integer  $n$  whose first digit is three, such that  $3n/2$  is obtained by removing the 3 at the beginning and putting it at the end.
114. (a) [1] Show that for any real  $x$ ,  $e^x > x$ .  
 (b) [1.5] Find the largest real number  $\alpha$  for which it is *false* that  $\alpha^x > x$  for all real  $x$ .
115. (a) [1] What amounts of postage cannot be obtained using only 5 cent stamps and 7 cent stamps? (For instance, 9 cents cannot be obtained, but  $17 = 5 + 5 + 7$  cents can be.)  
 (b) [2.5] Let  $a$  and  $b$  be relatively prime positive integers. For how many positive integers  $c$  is it impossible to obtain postage of  $c$  cents using only  $a$  cent and  $b$  cent stamps? What is the largest value of  $c$  with this property?
116. [2.5] Two players A and B play the following game. Fix a positive real number  $x$ . A and B each choose the number 1 or 2. A gives B one dollar if the numbers are different. B gives A  $x$  dollars times the sum of their numbers. For instance, if A chooses 1 and B chooses 2, then A gives B one dollar and B gives A  $3x$  dollars. Both players are playing their best possible strategy. What value of  $x$  makes the game fair, i.e., in the long run both players should break even?
117. [3] Given positive integers  $n$  and  $b$ , define the *total  $b$ -ary expansion*  $T_b(n)$  as follows: Write  $n$  as a sum of powers of  $b$ , with no power occurring more than  $b - 1$  times. (This is just the usual base  $b$  expansion of  $n$ .) For instance, if  $n = 357948$  and  $b = 3$ , then we get

$$3^{11} + 3^{11} + 3^7 + 3^6 + 3^6 + 3^2.$$

Now do the same for each exponent, giving

$$3^{3^2+1+1} + 3^{3^2+1+1} + 3^{3+3+1} + 3^{3+3} + 3^{3+3} + 3^{1+1}.$$

Continue doing the same for every exponent not already a  $b$  or 1, until finally only  $b$ 's and 1's appear. In the present case we get that  $T_3(357948)$  is the array

$$3^{3^{1+1}+1+1} + 3^{3^{1+1}+1+1} + 3^{3+3+1} + 3^{3+3} + 3^{3+3} + 3^{1+1}.$$

Now define a sequence  $a_0, a_1, \dots$  as follows. Choose  $a_0$  to be any positive integer, and choose a base  $b_0 > 1$ . To get  $a_1$ , write the total  $b_0$ -ary expansion  $T_{b_0}(a_0 - 1)$  of  $a_0 - 1$ , choose a base  $b_1 > b_0$ , and replace every appearance of  $b_0$  in  $T_{b_0}(a_0 - 1)$  by  $b_1$ . This gives the total  $b_1$ -ary expansion of the next term  $a_1$ . To get  $a_2$ , write the total  $b_1$ -ary expansion  $T_{b_1}(a_1 - 1)$  of  $a_1 - 1$ , choose a base  $b_2 > b_1$ , and replace every appearance of  $b_1$  in  $T_{b_1}(a_1 - 1)$  by  $b_2$ . This gives the total  $b_2$ -ary expansion of the next term  $a_2$ . Continue in this way to obtain  $a_3, a_4, \dots$ . In other words, given  $a_n$  and the previously chosen base  $b_n$ , to get  $a_{n+1}$ , write the total  $b_n$ -ary expansion  $T_{b_n}(a_n - 1)$  of  $a_n - 1$ , choose a base  $b_{n+1} > b_n$ , and replace every appearance of  $b_n$  in  $T_{b_n}(a_n - 1)$  by  $b_{n+1}$ . This gives the total  $b_{n+1}$ -ary expansion of the next term  $a_{n+1}$ .

**Example.** Choose  $a_0 = 357948$  and  $b_0 = 3$  as above. Then

$$a_0 - 1 = 357947 = 3^{3^{1+1}+1+1} + 3^{3^{1+1}+1+1} + 3^{3+3+1} + 3^{3+3} + 3^{3+3} + 3 + 3 + 1 + 1.$$

Choose  $b_1 = 10$ . Then

$$\begin{aligned} a_1 &= 10^{10^{1+1}+1+1} + 10^{10^{1+1}+1+1} + 10^{10+10+1} + 10^{10+10} + 10^{10+10} + 10 + 10 + 1 + 1 \\ &= 10^{102} + 10^{102} + 10^{21} + 10^{20} + 10^{20} + 22. \end{aligned}$$

Then

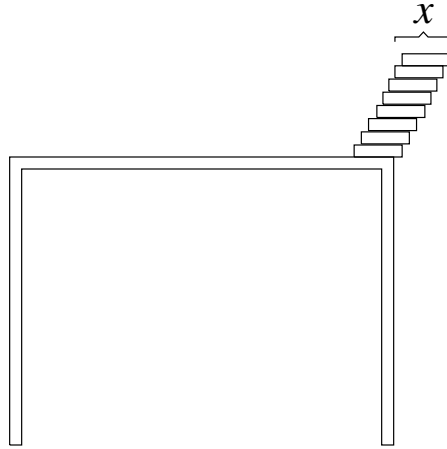
$$a_1 - 1 = 10^{10^{1+1}+1+1} + 10^{10^{1+1}+1+1} + 10^{10+10+1} + 10^{10+10} + 10^{10+10} + 10 + 10 + 1.$$

Choose  $b_2 = 766$ . Then

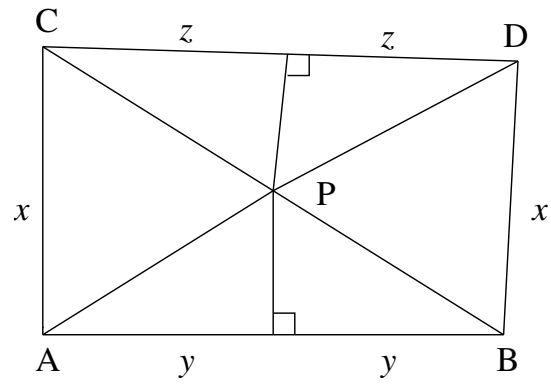
$$a_2 = 766^{766^{1+1}+1+1} + 766^{766^{1+1}+1+1} + 766^{766+766+1} + 766^{766+766} + 766^{766+766} + 766 + 766 + 1,$$

etc. Prove that for some  $n$  we have  $a_n = a_{n+1} = \dots = 0$ . (Note how counterintuitive this seems. How could we not force  $a_n \rightarrow \infty$  by choosing the  $b_n$ 's sufficiently large?)

118. [2.5] What is the longest possible overhang  $x$  that can be obtained by stacking dominos of unit length over the edge of a table, as illustrated below? (The condition for the dominos not to fall is that the center of mass of all the dominos above any domino  $D$  lies directly above  $D$ .)



119. [3] Let  $G$  be a simple (i.e., no loops or multiple edges) finite graph and  $v$  a vertex of  $G$ . The *neighborhood*  $N(v)$  of  $v$  consists of  $v$  and all adjacent vertices. Show that there exists a subset  $S$  of the vertex set of  $G$  such that  $\#(S \cap N(v))$  is odd for all vertices  $v$  of  $G$ .
120. [2.5] Let  $G$  be a simple graph with an odd number of vertices. Show that there exists a nonempty subset  $S$  of the vertices such that every vertex of  $G$  is adjacent to an even number of elements of  $S$ . (A vertex is *not* considered to be adjacent to itself.)
121. [2.5] An ant is constrained to walk on the walls, floor, and ceiling of a  $1 \times 1 \times 2$  room. The ant stands in a corner of the room. From its perspective, what point(s) in the room is the farthest away, and what is the distance of this point from the ant? HINT. The farthest point is *not* the opposite corner!
122. [2] Draw a line segment  $AB$  in the plane. Let  $AC$  be perpendicular to  $AB$ . Draw  $BD$  so that  $D$  is on the same side of  $AB$  as  $C$ , angle  $ABD$  is  $91^\circ$ , and  $AC$  and  $BD$  have the same length. Let the perpendicular bisectors of  $AB$  and  $CD$  meet in  $P$ . Angles  $PAB$  and  $PBA$  are equal since  $P$  is on the perpendicular bisector of  $AB$ . For the same reason  $AP$  and  $BP$  have the same length. Similarly  $CP$  and  $DP$  have the same length. Thus triangles  $ACP$  and  $BDP$  are congruent, so the angles  $CAP$  and  $DBP$  are equal. It follows that the angles  $CAB$  and  $DBA$  are equal, i.e.,  $90^\circ = 91^\circ$ . Is there a flaw in this argument, or is this the end of mathematics?



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