18.997 Topics in Combinatorial Optimization	May 11, 2004
Lecture 22	
Lecturer: Michel X. Goemans	*Scribe: Alantha Newman*

1 Multiflows and Disjoint Paths

Let $G = (V, E)$ be a graph and let $s_1, t_1, s_2, t_2, \ldots s_k, t_k \in V$ be terminals. Our goal is to find disjoint paths between s_i and t_i for each $i, 1 \leq i \leq k$. There are directed and undirected versions of this problem, i.e. G can be directed or undirected and we may want to find directed paths from s_i to t_i or undirected paths between these terminal pairs. Additionally, we specify if we want to find vertex disjoint paths or edge disjoint paths (arc disjoint paths for directed graphs). These disjoint path problems can be viewed as specific cases of the *multiflow problem*.

1.1 Multiflows

Suppose we are given the following inputs:

- a graph $G = (V, E)$ (directed or undirected),
- terminals $s_1, t_1, s_2, t_2, \ldots s_k, t_k \in V$,
- demands $d_i : i = 1, \ldots, k$,
- integer (or rational) capacities on the edges, $c : E \rightarrow \mathcal{Z}^+$.

For each i, find an (s_i, t_i)-flow f_i of value d_i. Note that even for undirected graphs, flow is directed. Let $f_i(e)$ be the amount of flow from s_i to t_i that uses edge e. A valid flow must obey the capacity constraint: for each edge $e \in E$, $\sum_{i=1}^{k} f_i(e) \leq c(e)$.

1.2 Edge Disjoint Paths

To find edge disjoint paths, we can set $c(e) = 1$ for all $e \in E$ and then find an integer multiflow. The problem of finding vertex disjoint paths in a directed graph can be reduced to the problem of finding edge disjoint paths in a directed graph; every vertex $v \in V$ undergoes the transformation shown in figure 1. Thus, a set of edge disjoint paths in the modified graph corresponds to a set of paths in the original graph in which each vertex is used at most once.

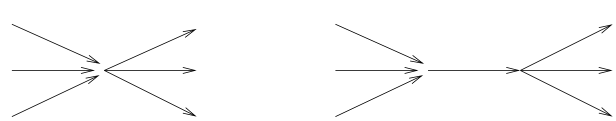

Figure 1: Each vertex undergoes the illustrated transformation.

Today, we focus on finding edge disjoint paths in undirected graphs. Note that the problem of finding edge disjoint paths is *very* different in terms of complexity for directed and undirected graphs.

Edge Disjoint Paths in Undirected Graphs: G is an undirected graph. Do there exist two edge disjoint paths between s and t? This problem can be solved easily by determining if the minimum s-t cut contains at least two edges.

Arc Disjoint Paths in Directed Graphs: G is a directed graph. Do there exist two arc disjoint paths, one from s to t and one from t to s? This problem is NP-hard!

The edge disjoint paths problem in undirected graphs can be reduced to the arc disjoint paths problem in directed graphs. Each edge in the original undirected graph is replaced by the gadget shown in figure 2.

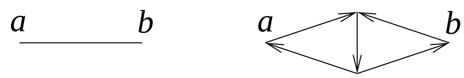

Figure 2: Each edge in the original undirected graph is replaced by the above gadget.

1.3 Fractional Multiflow

We focus on edge disjoint paths (multiflows) in undirected graphs. When $k = 1$, flow is easy. We can find integer flow using the max-flow min-cut theorem. In general, deciding if a multiflow exists can be determined by solving a linear program consisting of flow and capacity constraints.

Let $\mathcal{P}_i$ be set of all paths between s_i and t_i. We have a variable x_p for every such path $p \in \mathcal{P}_i$. We have the following primal LP:

$$\begin{aligned} \max\ 0 \cdot x & & \\ \sum_{p \in \mathcal{P}_i} x_p &= d_i \\ \sum_{i: p \in \mathcal{P}_i, e \in p} x_p &\leq c(e) \\ x_p &\geq 0. \end{aligned}$$

What does dual mean in this case? We use variables $\ell(e)$ for each edge $e \in E$, and variables b_i for $i = 1, \ldots, k$.

$$\begin{aligned} \min\ \sum_{e \in E} c(e)\ell(e) - \sum_{i=1}^{k} b_i d_i & & \qquad (1) \\ \sum_{e \in p} \ell(e) - b_i &\geq 0 \quad \forall p \in \mathcal{P}_i\ i = 1, \ldots k \\ \ell(e) &\geq 0. \end{aligned}$$

To make the term $(-\sum_{i=1}^{k} b_i d_i)$ small, we should make b_i as large as possible. Fix the edge function $\ell : E \to \mathcal{Q}$. Then b_i is the (minimum) $\text{dist}_\ell(s_i, t_i)$. The objective function of the dual (1) can be rewritten:

$$\sum_{e \in E} c(e)\ell(e) - \sum_{i=1}^{k} d_i\ dist_\ell(s_i, t_i).$$

If the primal LP is feasible, then there is no solution for the dual LP with a negative objective value. So there exists a fractional multiflow if and only if $\forall \ell(e) \geq 0, e \in E$, the following holds:

$$\sum_{e \in E} c(e)\ell(e) \geq \sum_{i=1}^{k} d_i \; dist_\ell(s_i, t_i). \tag{2}$$

Duality shows that this is a necessary and sufficient for the existence of a fractional multiflow.

2 Integer Multiflows

In general, the problem of determining when there is an integer multiflow is NP-complete. However, there are special conditions that imply the existence of an integer multiflow in certain classes of graphs.

Let R be a set of edges:

$$R = \{(s_i, t_i) : i = 1, \ldots k\}. \tag{3}$$

The set of edges in E outgoing from vertex set U is denoted by $\delta_E(U)$ and the set of edges in R outgoing from vertex set U is denoted by $\delta_R(U)$. A necessary condition for the existence of a multiflow (and thus of an integer multiflow) is the cut condition:

$$c(\delta_E(U)) \geq d(\delta_R(U)), \quad \forall U \subset V.$$

In general, the cut condition is not sufficient to guarantee the existence of an integer multiflow (or fractional multiflow) in a graph. However, in some cases of the multiflow problem, the cut condition is sufficient for the existence of a fractional multiflow. Furthermore, there are several cases known where the cut condition implies the existence of an integer multiflow when the *Euler condition* is satisfied:

$$c(\delta_E(v)) + d(\delta_R(v)) \text{ is even, for each vertex } v.$$

For example, when $k = 2$, we have the following implications:

(i) Cut condition $\Rightarrow$ fractional multiflow.

(ii) Cut condition and integer capacities $\Rightarrow$ half-integral multiflow.

(iii) Cut condition, integer capacities, and Euler condition $\Rightarrow$ integral multiflow.

The first proof of (i) and (ii) for the case when $k = 2$ was is due to Hu. The proof of (iii) is due to Rothschild and Winston. Note that (iii) implies (i) and (ii). For example, Consider the graph in Figure 3, let $d_1 = 1, d_2 = 1$. Let the capacity of each edge be 2. Note that the cut condition is satisfied but the Euler condition is not. However, suppose we double every capacity and demand, then the Euler condition is satisfied. We can convert an integer solution for this latter problem to a half-integral solution for the original problem.

Some "good" cases in which conditions (i), (ii) and (iii) are satisfied are:

1. If there are two commodities, i.e. $k = 2$, then cut condition and Euler condition are sufficient for integer multiflow.

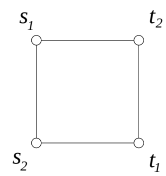

Figure 3: When the capacity of each edge in this graph is 2 and $d_1, d_2 = 1$, the Euler condition is not satisfied. There exists a half-integral multiflow, but no integral multiflow.

2. $G + D$ has no K_5 minor, e.g. $G + D$ is planar, where D is the demand graph, $D = (V, R)$ (see (3)).
3. $|\{(s_1, t_1), \ldots, (s_k, t_k)\}| \leq 4$.
4. G is planar and all (s_i, t_i) are on boundary of outside face. (Note that this does not imply case 2.)
5. If there are 2 faces and for each i, $(s_i, t_)$ are both on the inside face or both on the outside face.

3 Two-Commodity Flows

Theorem 1 (Rothschild and Whinston) *$G = (V, E)$ is an undirected graph such that $c(e) \in \mathcal{Z}^+$ for $e \in E$. Terminals s_1, t_1, s_2, t_2 are in V, and demands d_1, d_2 are positive integers. Additionally, the Euler condition is satisfied for G. Then G has an integer two-commodity flow if and only if the cut condition is satisfied.*

Proof: Our goal is to find flows from s_1 to t_1 and from s_2 to t_2 with values d_1 and d_2, respectively. We will show that if the cut condition is satisfied on G, then we can find such flows.

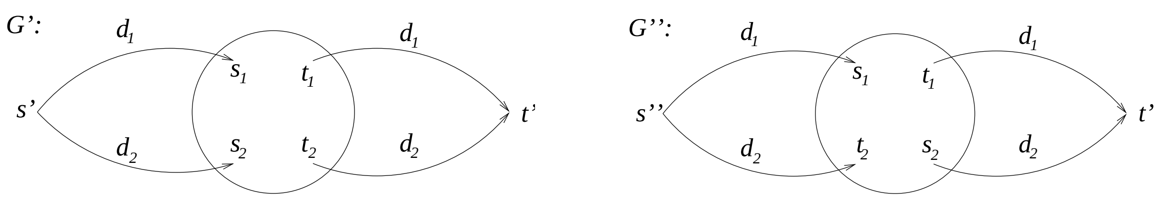

Figure 4: The graphs G' and G'' are constructed based on the given graph G.

First, based on the graph G, construct the graph G' as shown in figure 4. Let the edges (s', s_1) and (t_1, t') in G' have capacity d_1 and the edges (s', s_2) and (t_2, t') in G' have capacity d_2. By the max-flow min-cut theorem, we can find an integer s'-t' flow g with value $d_1 + d_2$, since the min-cut of G' has value $d_1 + d_2$. Note that this s'-t' flow does not necessarily give a two-commodity flow for the original problem (since some of the flow going through s_1 may end up in t_2).

Since the Euler condition is satisfied, we will prove that we can assume that $g(e) \equiv c(e) \mod 2$. To show this, first notice that the Euler condition implies that the total capacity incident to any vertex of G' is even. Furthermore, any integral flow will use up an even amount of capacity incident to any vertex. Now consider all the edges $e \in E$ such that $g(e) \not\equiv c(e) \mod 2$. Since it is the case that $\sum_{e\in\delta(v)}(g(e)-c(e))=0 \mod 2$, it follows that an even number of edges adjacent to vertex v have $g(e)\not\equiv c(e) \mod 2$. Thus, the edges such that $g(e)\not\equiv c(e) \mod 2$ make up an Eulerian graph (and do not contain the arcs incident to s' and t' that we added to G to make up G'). We can decompose this Eulerian graph into cycles, and push push one unit of flow across all these cycles (either increasing or decreasing the flow by one unit along it depending on the orientation), changing the parity of $g(e)$ for each such edge. Thus, for all edges $e\in E$, we have that $g(e)\equiv c(e) \mod 2$.

For G'', we have the same argument. Thus, we find an integer flow h in G'' with value d_1+d_2 such that $h(e)=c(e)$, $\forall e\in E$. Thus, for all edges $e\in E$, $h(e)=g(e) \mod 2$. We arbitrarily orient the edges of E to obtain A. So for all $a\in A, h(a)\equiv g(a) \mod 2$.

Now we define two flows on the graph G:

$$f_1(a)=\frac{1}{2}[g(a)+h(a)]$$
$$f_2(a)=\frac{1}{2}[g(a)-h(a)].$$

The following properties are true for the flows f_1 and f_2:

1. $f_1(a), f_2(a)$ are integer flows (since $f(a)$ and $g(a)$ have the same parity).
2. $|f_1(a)|+|f_2(a)|=\frac{1}{2}|g(a)+h(a)|+\frac{1}{2}|g(a)-h(a)|\leq \max(|g(a)|,|h(a)|)\leq c(a)$.
3. f_1 is d_1 units of flow from s_1 to t_1 and f_2 is d_2 units of flow from s_2 to t_2.

The last property holds because we can show that $f_1(\delta^+(s_1))-f_1(\delta^-(s_1))=d_1$ and $f_1(\delta^-(t_1))-f_1(\delta^+(t_1))=d_1$. By conservation of flow, if we consider the vertex s_1 in G, we have:

$$g(\delta^+(s_1))-g(\delta^-(s_1))=d_1 \tag{4}$$
$$h(\delta^+(s_1))-h(\delta^-(s_1))=d_1 \tag{5}$$

Equations (4) and (5) imply $f_1(\delta^+(s_1))-f_1(\delta^-(s_1))=d_1$.

$$g(\delta^-(t_1))-g(\delta^+(t_1))=d_1 \tag{6}$$
$$h(\delta^-(t_1))-h(\delta^+(t_1))=d_1. \tag{7}$$

Equations (6) and (7) imply $f_1(\delta^-(t_1))-f_1(\delta^+(t_1))=d_1$. Similarly, we can show that the last property holds for flow f_2. If we consider vertices s_2 and t_2 in G, we have:

$$g(\delta^+(s_2))-g(\delta^-(s_2))=d_2 \tag{8}$$
$$h(\delta^-(s_2))-h(\delta^+(s_2))=d_2 \tag{9}$$
$$g(\delta^-(t_2))-g(\delta^+(t_2))=d_2 \tag{10}$$
$$h(\delta^+(t_2))-h(\delta^-(t_2))=d_2. \tag{11}$$

Equations (8) and (9) imply $f_2(\delta^+(s_2))-f_2(\delta^-(s_2))=d_2$ and equations (10) and (11) imply $f_2(\delta^-(t_2))-f_2(\delta^+(s_2))=d_2$. □

As a final note, consider the problem of maximizing the sum of the flow between s_1 and t_1 and between s_2 and t_2. This is the *max biflow problem.* A *bicut* is a cut separating s_1 from t_1 and s_2 from t_2, thus it is either a cut separating s_1, s_2 from t_1, t_2 or a cut separating s_1, t_2 from s_2, t_1. One can show that the following theorem follows from Theorem 1.

Theorem 2 *The maximum biflow equals the minimum bicut.*