SYMPLECTIC GEOMETRY, LECTURE 7

Prof. Denis Auroux

1. Floer homology

For a Hamiltonian diffeomorphism $f : (M, \omega) \to (M, \omega)$, $f = \phi^1_H$, $H_t : M \to \mathbb{R}$ 1-periodic in t, we want to look for fixed points of f, i.e. 1-periodic orbits of X_H, $x'(t) = X_{H_t}(x(t))$. We consider the Floer complex $CF^*(f)$, whose basis are 1-periodic orbits; these correspond to critical points of the action functional $\mathcal{A}_H$ on a covering of the free loop space $\Omega(M)$. The differential 'counts' solutions of Floer's equations

$$u : \mathbb{R} \times S^1 \to M,\ \frac{\partial u}{\partial s} + J(u(s,t))(\frac{\partial u}{\partial t} - X_{H_t}(u)) = 0 \tag{1}$$

such that $\lim_{s\to\pm\infty} u(s, \cdot) = x_\pm$ (1-periodic orbits). The solutions are formal gradient flow lines of $\mathcal{A}_H$ between the critical points $x_\pm$.

Theorem 1 (Arnold's conjecture)**.** *If the fixed points of f are nondegenerate, then $\#\mathrm{Fix}(f) \geq \sum_i \dim H^i(M)$, i.e. $\#\mathrm{Fix}(f) = \mathrm{rk}\ CF^* \geq \mathrm{rk}\ HF^* = \mathrm{rk}\ H^*(CF^*, \partial) = \mathrm{rk} H^*(M)$.*

1.1. Lagrangian intersections.

There is a notion of Lagrangian Floer homology, which is not always defined (in fact, there are explicit obstructions to its existence). The idea is to count intersections of Lagrangian submanifolds $L, L' \subset M$ in a manner which is invariant under Hamiltonian deformations (isotopies). Assume that L and L' are transverse (if not, e.g. when $L = L'$, replace the submanifold L by the graph L_t of an exact 1-form in T^*L). To define Floer homology, one defines a complex $CF^*(L, L')$ whose basis is the set of intersection points, and whose differential is given by $\partial p = \sum_q n_{p,q} q$, where $n_{p,q}$ counts solutions to

$$u : \mathbb{R} \times [0, 1] \to M, u(\mathbb{R} \times 0) \subset L, u(\mathbb{R} \times 1) \subset L', \frac{\partial u}{\partial s} + J\frac{\partial u}{\partial t} = 0 \tag{2}$$

Under suitable assumptions, one finds that $\partial^2 = 0$, giving us a Floer homology

$$HF^*(L, L') = H^*(CF^*(L, L'), \partial) \tag{3}$$

which is invariant under Hamiltonian deformations of L, L'. Moreover, $\mathrm{rk}\ HF^* \leq \mathrm{rk}\ CF^* = |L \cap L'|$.

Theorem 2 (Floer, Oh, Fukaya-Oh-Ohta-Ono)**.** *Given a compact Lagrangian submanifold $L \subset M$ which is "relatively spin" (i.e. $w_2(TL) \in \mathrm{Im}\{i^* : H^2(M, \mathbb{Z}/2\mathbb{Z}) \to H^2(L, \mathbb{Z}/2\mathbb{Z})\}$) s.t. $i_* : H_1(L, \mathbb{Q}) \to H_1(M, \mathbb{Q})$ is injective, then $\forall \psi \in \mathrm{Ham}(M, \omega)$ s.t. $\psi(L)$ intersects L transversely, $\#(L \cap \psi(L)) \geq \sum \dim H_i(L, \mathbb{Q})$.*

Remark. Applying this theorem to the diagonal $\Delta = \Delta(M) \subset M \times M$ and the graph of a Hamiltonian diffeomorphism f on M, one recovers Arnold's conjecture.

2. Almost-Complex Structures

To begin, we will study complex structures on vector spaces.

Definition 1. *A* complex structure *on a vector space V is an endomorphism $J : V \to V$ s.t. $J^2 = -I$. Thinking of this J as multiplication by i turns V into a complex vector space, $(x + iy)v = xv + yJv$. If V is a symplectic vector space with symplectic form Ω, a complex structure is* compatible *if $G(u, v) = \Omega(u, Jv)$ is a positive symmetric inner product. Note that being symmetric is equivalent to $\Omega(Ju, Jv) = \Omega(u, v)$, and being positive is precisely $\Omega(u, Ju) > 0$ $\forall u \neq 0$.*

Example. Let $V = (\mathbb{R}^{2n}, \Omega_0)$ be the standard symplectic vector space, with standard basis $e_1, \dots, e_n, f_1, \dots, f_n$, and define J_0 by $e_i \mapsto f_i, f_i \mapsto -e_i$. Then

$$J_0^2 = -\mathrm{id}\ , G_0(u,v) = \Omega_0(u, J_0 v) \implies G_0(e_i, e_i) = 1, G_0(f_i, f_i) = 1 \tag{4}$$

and all other pairings are 0. In matrix terms, $\Omega_0 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$, and $J_0 = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$, so $G_0 = \Omega_0 J_0 = I$. This gives us a natural isomorphism with $\mathbb{C}^n$.

Proposition 1. *If (V, Ω) is a symplectic vector space, $\exists$ a compatible J. Moreover, given any positive inner product $\langle\cdot,\cdot\rangle$ on V, we can build an Ω-compatible complex structure on V canonically (though it has no direct relation to the given inner product).*

Proof. For the first part, taking $J = J_0$ in a standard basis gives the desired endomorphism. For the second part, by the nondegeneracy of Ω, we have isomorphisms $u \mapsto \Omega(u, \cdot)$ and $u \mapsto \langle u, \cdot\rangle$ from V to V^*. We thus obtain an endomorphism $A = \langle\rangle^{-1} \circ \Omega$ s.t. $\Omega(u,v) = \langle Au, v\rangle$. A is invertible and skew-symmetric w.r.t. $\langle\rangle$, i.e. $A^* = -A$ (since $\Omega(v,u) = \langle Av, u\rangle = \langle v, A^*u\rangle = \langle A^*u, v\rangle = -\Omega(u,v) = -\langle Au, v\rangle$). Thus, $AA^* = -A^2$ is symmetric and positive definite, therefore diagonalizable with real, strictly positive eigenvalues. This implies the existence of a square root $\sqrt{AA^*}(= \mathrm{diag}(\sqrt{\lambda_i}))$, so define $J = (\sqrt{AA^*})^{-1}A$. (Note that the decomposition $A = \sqrt{AA^*}J$ gives a "polar decomposition" of A.) A commutes with $\sqrt{AA^*}$: letting V_i be the eigenspace of AA^* with eigenvalue λ_i, or similarly that of $\sqrt{AA^*}$ with eigenvalue $\sqrt{\lambda_i}$, we find that,

$$\forall v \in V_i, (AA^*)Av = -A^3 v = A(AA^*)v = \lambda_i Av \implies Av \in V_i \tag{5}$$

So J also commutes with A and with $\sqrt{AA^*}$, and thus is skew-symmetric

$$J^* = A^*(\sqrt{AA^*})^{-1} = -A(\sqrt{AA^*})^{-1} = -J \tag{6}$$

and orthogonal

$$J^*J = A^*(\sqrt{AA^*})^{-1}(\sqrt{AA^*})^{-1}A = \mathrm{id} \tag{7}$$

In particular, $J^2 = -J^*J = -\mathrm{id}$. For compatibility, note that

$$\begin{aligned} \Omega(Ju, Jv) &= \langle AJu, Jv\rangle = \langle JAu, Jv\rangle = \langle Au, v\rangle = \Omega(u,v) \\ \Omega(u, Ju) &= \langle Au, Ju\rangle = \langle -JAu, u\rangle = \langle -(\sqrt{AA^*})^{-1}AAu, u\rangle \\ &= \langle (\sqrt{AA^*})^{-1}(AA^*)u, u\rangle = \langle (\sqrt{AA^*})u, u\rangle > 0 \end{aligned} \tag{8}$$

thus completing the proof. □

Remark. Note that $G(u,v) = \Omega(u, Jv) = \langle\sqrt{AA^*}u, v\rangle$, so if $\langle\cdot,\cdot\rangle$ was already compatible with Ω, then $AA^* = I, J = A, G = \langle\cdot,\cdot\rangle$.

Definition 2. *An* almost-complex structure *on a manifold M is $J \in \mathrm{End}(TM)$ s.t. $J^2 = -I$ (i.e. $\forall x \in M, J_x$ is a complex structure on T_xM). If $M = (M, \omega)$ is a symplectic manifold, J is* compatible *if $\forall x \in M, J_x$ is ω_x-compatible, with associated Riemannian metric $g_x(u,v) = \omega_x(u, J_x v)$. We say that (ω, g, J) is a* compatible triple*, with any two determining the third.*