18.917 Topics in Algebraic Topology: The Sullivan Conjecture
Fall 2007

Analytic Functors Revisited (Lecture 35)

In this lecture, we will revisit the relationship between unstable modules over the (mod 2) Steenrod algebra $\mathcal{A}$ and analytic functors from the category of $\mathbf{F}_2$ vector spaces to itself.

Let Vect denote the category of $\mathbf{F}_2$-vector spaces, Vect^f the full subcategory consisting of finite dimensional vector spaces, and Fun the category of all functors from Vect^f to Vect. Recall that $\mathrm{Fun}^{\mathrm{an}}$ denotes the category of *analytic functors*: that is, functors which can be obtained as colimits of functors $F : \mathrm{Vect}^f \to \mathrm{Vect}$ having the property that the function

$$n \mapsto \dim F(\mathbf{F}_2^n)$$

is a polynomial. In particular, $\mathrm{Fun}^{\mathrm{an}}$ contains the divided power functors Γ^n defined by the formula

$$\Gamma^n(V) = (V^{\otimes n})^{\Sigma_n}.$$

Let $\mathcal{U}$ denote the category of unstable modules over the Steenrod algebra. In a previous lecture, we studied a pair of adjoint functors

$$\mathcal{U} \underset{g}{\overset{f}{\rightleftarrows}} \mathrm{Fun}^{\mathrm{an}}.$$

This adjunction was essentially uniquely determined by the requirement that f carries a free unstable module $F(n)$ to the analytic functor Γ^n. We begin by reformulating this construction using Lannes' T-functor.

Let M be an unstable $\mathcal{A}$-module. For every $\mathbf{F}_2$-vector space V, the $\mathcal{A}$-module $T_V M$ is defined by the universal property

$$\mathrm{Hom}(T_V M, N) \simeq \mathrm{Hom}(M, N \otimes \mathrm{H}^*(BV)).$$

In particular, given a map $V \to W$, the composition

$$M \to T_W M \otimes \mathrm{H}^*(BW) \to T_W M \otimes \mathrm{H}^*(BV)$$

is classified by a map $T_V M \to T_W M$. In other words, $T_V M$ is a covariant functor of V.

Proposition 1. *The functor $f : \mathcal{U} \to \mathrm{Fun}^{\mathrm{an}}$ is defined by the formula*

$$f(M)(V) = (T_V M)^0.$$

Proof. This formula evidently defines a colimit-preserving functor from $\mathcal{U}$ to $\mathrm{Fun}^{\mathrm{an}}$. It is therefore determined by its values on free unstable $\mathcal{A}$-modules (since any module admits a free resolution). We will show that the above formula has the correct behavior on objects, and leave to the reader to check that the behavior on morphisms is correct. For this, we compute

$$\begin{aligned}
(T_V F(n))^0 &\simeq \mathrm{Hom}(T_V F(n), \mathbf{F}_2)^\vee \\
&\simeq \mathrm{Hom}(F(n), \mathrm{H}^*(BV))^\vee \\
&\simeq \mathrm{H}^n(BV)^\vee \\
&\simeq \mathrm{Sym}^n(V^\vee)^\vee \\
&\simeq \Gamma^n V.
\end{aligned}$$

□

From this description and the exactness of T_V, we immediately deduce that the functor $f : \mathcal{U} \to \mathrm{Fun}^{\mathrm{an}}$ is exact. Of course, this reasoning is circular: earlier, we used the exactness of f to prove that $\mathrm{H}^*(BV)$ was an injective object of $\mathcal{U}$, which was a key step in the proof that the functor T_V is exact.

We now wish to generalize the above construction. We first expand on the observation that $T_V M$ depends functorially on V. Fix an integer $n \geq 0$. We will say that an unstable $\mathcal{A}$-module M is *n-truncated* if $M^i = 0$ for $i > n$. Given any unstable $\mathcal{A}$-module M, we can define an n-truncated $\mathcal{A}$-module $\tau^{\leq n} M$ by the formula

$$(\tau^{\leq n} M)^i = \begin{cases} M^i & \text{if } i \leq n \\ 0 & \text{otherwise.} \end{cases}$$

In other words, $\tau^{\leq n} M$ is the quotient of M obtained by killing all elements of degree larger than n. The collection of all n truncated unstable $\mathcal{A}$-modules forms a category which we will denote by $\mathcal{U}^{\leq n}$. This category inherits a symmetric monoidal structure $\boxtimes$, given by the formula

$$M \boxtimes N \mapsto \tau^{\leq n}(M \otimes N),$$

where $\otimes$ denotes the usual tensor product of $\mathcal{A}$-modules.

We now define a category $\mathcal{C}_n$ which is enriched over the *opposite* of $\mathcal{U}^{\leq n}$, as follows:

- The objects of $\mathcal{C}_n$ are finite dimensional $\mathbf{F}_2$-vector spaces V.
- Given a pair of objects V and W, we have

$$\mathrm{Map}_{\mathcal{C}}(V, W) = T_V \, \mathrm{H}^*(BW) \simeq \mathrm{H}^*(BW)^{BV}.$$

- Composition in $\mathcal{C}_n$ is induced by the maps

$$(BW)^{BV} \times (BV)^{BU} \to (BW)^{BU}.$$

We let Fun_n denote the category consisting of all $\mathcal{U}^{\leq n, op}$-enriched functors from $\mathcal{C}_n$ to $\mathcal{U}^{\leq n}$. In other words, an object F of Fun_n can be described as follows:

- For every finite dimensional $\mathbf{F}_2$-vector space V, $F(V)$ is an n-truncated unstable $\mathcal{A}$-module.
- For every pair of $\mathbf{F}_2$-vector spaces V and W, we have an associated map of $\mathcal{A}$-modules

$$F(V) \to \tau^{\leq n}(T_V \, \mathrm{H}^*(BW) \otimes F(W)).$$

- These maps are compatible with composition in the obvious sense.

Example 2. Let M be an unstable $\mathcal{A}$-module, and define $P_M(V)$ by the formula

$$P_M(V) = \tau^{\leq n} T_V(M).$$

For every pair of $\mathbf{F}_2$-vector spaces V and W, the canonical map

$$M \to T_W M \otimes \mathrm{H}^*(BW) \to T_W M \otimes T_V \, \mathrm{H}^*(BW) \otimes \mathrm{H}^*(BV)$$

is adjoint to a map

$$T_V M \to T_W M \otimes T_V \, \mathrm{H}^*(BW).$$

Truncating, we obtain a map

$$P_M(V) \to \tau^{\leq n}(P_M(W) \otimes T_V \, \mathrm{H}^*(BW)),$$

so that P_M can be viewed as an object of Fun_n.

Example 3. Suppose $n = 0$. An n-truncated $\mathcal{A}$-module M can be identified with its underlying $\mathbf{F}_2$-vector space M^0. An object $F \in \mathrm{Fun}_0$ associates to each $\mathbf{F}_2$-vector space V a new vector space $F(V)$, and to each pair (V, W) a map

$$F(V) \to F(W) \otimes \mathrm{H}^0(BW)^{BV} \simeq F(W) \otimes \mathbf{F}_2^{\mathrm{Hom}(V,W)}.$$

This is equivalent to giving a map $F(V) \to F(W)$ for every map of vector spaces from V to W. In other words, we can identify F with a functor from Vect^f to Vect. Consequently, Fun_0 is canonically equivalent to the category Fun defined above.

Remark 4. More generally, for any $n \geq 0$ and any $F \in \mathrm{Fun}_n$, we have canonical maps

$$F(V) \to F(W) \otimes \mathrm{H}^0(BW)^{BV} \simeq F(W) \otimes \mathbf{F}_2^{\mathrm{Hom}(V,W)}.$$

which allow us to view $F(V) \in \mathcal{U}$ as a covariant functor of V. We will say that F is *analytic* (polynomial, etcetera) if this underlying functor is analytic. Let $\mathrm{Fun}_n^{\mathrm{an}}$ denote the full subcategory of Fun_n consisting of analytic functors.

The construction $M \mapsto P_M$ defines a functor

$$f_n : \mathcal{U} \to \mathrm{Fun}_n .$$

In the special case $n = 0$, we recover the functor studied earlier in this course. We now generalize some of our previous results:

Proposition 5. *Let $n \geq 0$.*

(1) *For every unstable $\mathcal{A}$-module M, the functor $f_n M \in \mathrm{Fun}_n$ is analytic.*

(2) *The functor f_n determines an adjunction*

$$\mathcal{U} \underset{g_n}{\overset{f_n}{\rightleftarrows}} \mathrm{Fun}_n^{\mathrm{an}} .$$

(3) *The functor f_n is exact.*

(4) *The functor g_n is fully faithful.*

Proof. To prove (1), it suffices to treat the case where M is a free unstable module $F(k)$. In this case, we have we will prove the following stronger assertion:

(1′) The functor $f_n F(k) = P_{F(k)}$ is polynomial and each $P_{F(k)}(V)^i$ is finite dimensional.

To prove this, we simply compute

$$\begin{aligned} (f_n F(k))(V)^i &\simeq (T_V F(k))^i \\ &\simeq \mathrm{Hom}(T_V F(k), J(i))^\vee \\ &\simeq \mathrm{Hom}(F(k), J(i) \otimes \mathrm{H}^*(BV))^\vee \\ &\simeq \oplus_{k=k'+k''} (J(i)^{k'})^\vee \otimes \Gamma^{k'}(V). \end{aligned}$$

Assertion (2) follows from the adjoint functor theorem. Moreover, assertion (1′) yields a little bit more:

(2′) The functor g_n preserves filtered colimits.

To see this, we observe that for every integer i, we have

$$\begin{aligned}(g_n \varinjlim G_\alpha)^i &\simeq \mathrm{Hom}_{\mathcal{U}}(F(i), g_n \varinjlim G_\alpha)\\ &\simeq \mathrm{Hom}_{\mathrm{Fun}_n^{\mathrm{an}}}(f_n F(i), \varinjlim G_\alpha)\\ &\simeq \varinjlim \mathrm{Hom}_{\mathrm{Fun}_n^{\mathrm{an}}}(f_n F(i), G_\alpha)\\ &\simeq \varinjlim \mathrm{Hom}_{\mathcal{U}}(F(i), g_n G_\alpha)\\ &\simeq \varinjlim (g_n G_\alpha)^i\end{aligned}$$

Assertion (3) follows from the exactness of Lannes' T-functor. To prove (4), we need to introduce a bit of notation. For $0 \leq i \leq n$, let $I_{W,J(i)}$ denote the object $f_n(J(i) \otimes \mathrm{H}^*(BW)) \in \mathrm{Fun}_n$. Since T_V commutes with products and carries $J(i)$ to itself, we have

$$I_{W,J(i)}(V) = \tau^{\leq n}(J(i) \otimes T_V \, \mathrm{H}^*(BW)).$$

Using Yoneda's lemma, we deduce the existence of a canonical isomorphism

$$\mathrm{Hom}_{\mathrm{Fun}_n}(F, I_{W,J(i)}) = \mathrm{Hom}_{\mathcal{U}}(F(W), J(i)).$$

In particular $I_{W,J(i)}$ is injective in Fun_n. We claim that $I_{W,J(i)}$ is analytic. To prove this, it suffices to show that for $j \leq n$ the functor

$$V \mapsto \oplus_{j=j'+j''} J(i)^{j'} \otimes \mathrm{H}^{j''}(BW)^{BV}$$

is analytic. For this, it suffices to show that the functor

$$V \mapsto \mathrm{H}^{i''}(BW)^{BV}$$

is analytic. This functor is a summand of the functor

$$V \mapsto \mathrm{H}^*(BW)^{BV} \simeq \mathrm{H}^*(BW) \otimes \mathbf{F}_2^{\mathrm{Hom}(V,W)}.$$

The first factor is constant, and the second factor was shown to be analytic in a previous lecture.

Let M be an unstable $\mathcal{A}$-module. We compute

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Fun}_n}(f_n M, I_{W,J(i)}) &\simeq \mathrm{Hom}_{\mathcal{U}}((f_n N)(W), J(i))\\ &\simeq \mathrm{Hom}_{\mathcal{U}}(\tau^{\leq n} T_W M, J(i))\\ &\simeq \mathrm{Hom}_{\mathcal{U}}(T_W M, J(i))\\ &\simeq \mathrm{Hom}_{\mathcal{U}}(M, J(i) \otimes \mathrm{H}^*(BW)).\end{aligned}$$

In other words, we can identify $g_n I_{W,J(i)}$ with $J(i) \otimes \mathrm{H}^*(BW)$. It follows that the unit map $f_n g_n \to \mathrm{id}$ is an isomorphism when evaluated on $I_{W,J(i)}$.

Every object $F \in \mathrm{Fun}_n^{\mathrm{an}}$ can be written as a union of its finitely generated subfunctors, which are polynomial functors of finite type and therefore have finite length as objects of $\mathrm{Fun}_n^{\mathrm{an}}$. It follows that $\mathrm{Fun}_n^{\mathrm{an}}$ is a locally Noetherian abelian category in which every Noetherian object has finite length. It follows that the indecomposable injective objects of $\mathrm{Fun}_n^{\mathrm{an}}$ are precisely the injective hulls of the simple objects. Let F be simple, and let I be an injective hull of F. Then for some vector space W, we have $F(W) \neq 0$ so there exists a nontrivial map $F(W) \to J(i)$ for $0 \leq i \leq n$. This classifies a nonzero map $F \to I_{W,J(i)}$. Since $I_{W,J(i)}$ is injective, we can extend this to a map $\phi : I \to I_{W,J(i)}$. The kernel of this map does not intersect $F \subseteq I$, and is therefore itself zero (since I is an injective hull of F). It follows that ϕ is a monomorphism between injective objects of $\mathrm{Fun}_n^{\mathrm{an}}$, so that ϕ splits. In other words, every indecomposable injective can be obtained as a direct summand of some $I_{W,J(i)}$. Since *every* injective object of $\mathrm{Fun}_n^{\mathrm{an}}$ can be written as a direct sum of indecomposable injectives (this is true in any Grothendieck abelian category), we conclude that every injective can be obtained as a summand of an expression of the form $\oplus_\alpha I_{W_\alpha, J(i_\alpha)}$.

It follows that any functor $G \in \mathrm{Fun}_n^{\mathrm{an}}$ admits an injective resolution

$$0 \to G \to \oplus_\alpha I_{W_\alpha, J(i_\alpha)} \to \oplus_\beta I_{W_\beta, J(i_\beta)}$$

Since f_n and g_n are both left exact, we get a diagram of short exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & G & \longrightarrow & \oplus_\alpha I_{W_\alpha, J(i_\alpha)} & \longrightarrow & \oplus_\beta I_{W_\beta, J(i_\beta)} \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & g_n f_n G & \longrightarrow & g_n f_n \oplus_\alpha I_{W_\alpha, J(i_\alpha)} & \longrightarrow & g_n f_n \oplus_\beta I_{W_\beta, J(i_\beta)} \end{array}$$

To prove that the left vertical arrow is an isomorphism, it suffices to show that the other two vertical arrows are isomorphisms. Since f_n and g_n both commute with direct sums, we can reduce to the case where $G = I_{W, J(i)}$, which was handled above. □