18.917 Topics in Algebraic Topology: The Sullivan Conjecture
Fall 2007

Topics in Algebraic Topology (18.917): Lecture 23

In this lecture, we will discuss the convergence of the cohomological Eilenberg-Moore spectral sequence. We begin with a definition.

Definition 1. Let p be a prime number. A topological space X is *p-finite* if the following conditions are satisfied:

- The set $\pi_0 X$ is finite.
- For every point $x \in X$ and every $i > 0$, the group $\pi_i(X, x)$ is a finite p-group.
- The groups $\pi_i(X, x)$ vanish for $i \gg 0$.

Example 2. Every Eilenberg-MacLane space of the form $K(\mathbf{Z}/p^k\mathbf{Z}, n)$ is p-finite.

Remark 3. Suppose given a fibration $f : E \to B$, where B is p-finite. Then E is p-finite if and only if each fiber of f is p-finite (this follows from the long exact sequence of homotopy groups).

Lemma 4. *A path connected topological space X is p-finite if and only if there exists a sequence of fibrations*

$$X \simeq X_m \to X_{m-1} \to \ldots \to X_0 \simeq *$$

where each X_i is a principal fibration over X_{i-1} with fiber $K(\mathbf{F}_p, j)$ for some integer $j \geq 1$.

Proof. The "if" direction follows from Remark 3. To prove the converse, we work by induction on $p^k = \prod_i |\pi_i(X, x)|$, where x is some fixed base point of X. If $k = 0$ then X is weakly contractible and there is nothing to prove. Otherwise, there exists some largest $i > 0$ such that $\pi_i(X, x)$ does not vanish.

Each orbit of $\pi_1(X, x)$ on $\pi_i(X, x)$ has cardinality a power of p, and the sum of the cardinality of the orbits is again a power of p. Since there is an orbit of size 1 (the orbit of the identity element), there must be at least p orbits of size 1: in other words, the subgroup G of $\pi_1(X, x)$-invariants in $\pi_i(X, x)$ is nontrivial. Since G is a finite p-group, there exists an element of G of order p; let G_0 be the cyclic subgroup of order p. Let X' be the space obtained from X by killing the subgroup $G_0 \subseteq \pi_i(X, x)$. Then $X \to X'$ is equivalent to a principal fibration with fiber $K(\mathbf{F}_p, i)$. We now conclude by applying the inductive hypothesis to X'. $\square$

Corollary 5. *Let X be a p-finite space. Then each cohomology group $\mathrm{H}^n(X; \mathbf{F}_p)$ is a finite dimensional vector space over $\mathbf{F}_p$.*

Proof. The result is true when $X = K(\mathbf{F}_p, n)$ by an explicit calculation (which we performed in a previous lecture when $p = 2$). The result follows in general from Lemma 4 and the Serre spectral sequence. $\square$

The main result of today's lecture is the following:

Theorem 6. *Suppose given a homotopy pullback square*

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d] \\ Y' \arrow[r] & Y \end{tikzcd}$$

of p-finite spaces. Then the induced square

$$\begin{tikzcd} C^*(X';\mathbf{F}_p) & C^*(X;\mathbf{F}_p) \arrow[l] \\ C^*(Y';\mathbf{F}_p) \arrow[u] & C^*(Y;\mathbf{F}_p) \arrow[l] \arrow[u] \end{tikzcd}$$

is a homotopy pushout square of E_∞-algebras.

Remark 7. The proof of Theorem 6 really requires much weaker hypotheses than p-finiteness, but this version will be sufficient for our immediate needs.

For the remainder of this lecture, we let $C^*(Z)$ denote the mod-p cochain complex $C^*(Z;\mathbf{F}_p)$ of a topological space Z. Theorem 6 asserts that the canonical map

$$C^*(X) \otimes_{C^*(Y)} C^*(Y') \to C^*(X')$$

induces an isomorphism after passing to cohomology. In the case where Y is a point, we can identify $C^*(Y)$ with $\mathbf{F}_p$; then Theorem 6 follows from the Kunneth theorem (since $\mathrm{H}^*(X)$ and $\mathrm{H}^*(Y')$ are finite dimensional in each degree thanks to Corollary 5).

In general, it is natural to try to prove Theorem 6 using a relative version of the same argument. For each point $y \in Y$, let X_y, X'_y, and Y'_y denote the (homotopy) fibers of X, X', and Y' over the point y. We then have an identification $X'_y \simeq X_y \times Y'_y$, which induces an equivalence of E_∞-algebras

$$C^*(X_y) \otimes C^*(Y'_y) \to C^*(X'_y).$$

The E_∞-algebras $C^*(X'_y)$ and $C^*(X'_{y'})$ are equivalent whenever y and y' lie in the same path component of Y, and are *canonically* equivalent if we specify a path from y to y' (since the choice of such a path induces a weak homotopy equivalence of fibers $X'_y \simeq X'_{y'}$). In other words, we can regard the construction

$$y \mapsto C^*(X'_y)$$

as providing a *local system* L of E_∞-algebras over Y. Moreover, we can identify $C^*(X')$ with the cochain complex $C^*(Y;L)$ of Y with coefficients in L. Similarly, we have local systems

$$L_0 : y \mapsto C^*(X_y)$$

$$L_1 : y \mapsto C^*(Y'_y).$$

and equivalences $C^*(X) \simeq C^*(Y;L_0)$, $C^*(Y') \simeq C^*(Y;L_1)$. The Kunneth theorem provides an equivalence $L \simeq L_0 \otimes L_1$ of local systems on Y. Theorem 6 then reduces to a special case of the following result:

Theorem 8. *Let Y be a p-finite space. Let L_0 and L_1 be local systems (of cochain complexes of $\mathbf{F}_p$-vector spaces) on Y satisfying the following condition:*

$(*)$ *The cohomology groups $\mathrm{H}^* L_0$ and $\mathrm{H}^* L_1$ vanish for $* < 0$.*

Then the canonical map

$$C^*(Y;L_0) \otimes_{C^*(Y)} C^*(Y;L_1) \to C^*(Y;L_0 \otimes L_1)$$

is an isomorphism on cohomology.

Let us say that a local system (of cochain complexes) L_0 on Y is *good* if it satisfies $(*)$, and the conclusion of Theorem 8 is satisfied for L_0 (and for any other local system L_1 satisfying $(*)$). We wish to show that every L_0 which satisfies $(*)$ is good.

For every local system L_0, we can define a new local system $\tau^{\leq n} L_0$ equipped with a map $\tau^{\leq n} L_0 \to L_0$, uniquely determined (up to quasi-isomorphism) by the following condition:

$$\mathrm{H}^k\, \tau^{\leq n} L_0 \simeq \begin{cases} \mathrm{H}^k\, L_0 & \text{if } k \leq n \\ 0 & \text{otherwise.} \end{cases}$$

Then L_0 is equivalent to the filtered colimit $\operatorname{inj\,lim}\{\tau^{\leq n} L_0\}$. To prove that L_0 is good, it will therefore suffice to show that each $\tau^{\leq n} L_0$ is good. In other words, we may assume that L_0 has cohomology only in finitely many degrees.

The collection of good local systems is also closed under extensions. We may therefore suppose that L is concentrated in a single degree, corresponding to a representation V of the fundamental group $\pi_1 Y$ (in some degree). Since $\pi_1 Y$ is finite, we can write V as a filtered colimit of finite-dimensional representations of $\pi_1 Y$. It therefore suffices to prove the result when V is finite dimensional, and we work by induction on the dimension of V. If $V \simeq 0$ there is nothing to prove. Assume that V is of positive dimension. The counting argument used in the proof of Lemma 4 shows that V contains a one-dimensional subspace $V_0 \subseteq V$ on which $\pi_1 Y$ acts trivially. By the inductive hypothesis, the local system V/V_0 is good. It will therefore suffice to show that V_0 is good. In other words, we have reduced the proof of Theorem 8 to the case where the local system L_0 is trivial.

Using the same argument, we can reduce to the case where L_1 is trivial. We can now restate Theorem 8 as the assertion that the canonical map

$$C^*(Y) \otimes_{C^*(Y)} C^*(Y) \to C^*(Y)$$

is an isomorphism on cohomology, which is obvious.

We conclude with an explanation of the relationship of Theorem 6 with the convergence of the Eilenberg-Moore spectral sequence. Let A be an E_∞-algebra, and let M and N be A-modules. Choosing a resolution of M or N (or both) by free modules, we obtain a spectral sequence for computing cohomology $\mathrm{H}^*(M \otimes_A N)$, with E_2-term given by

$$E_2^{p,q} = \mathrm{Tor}_{-p}^{\mathrm{H}^* A}(\mathrm{H}^*\, M \otimes \mathrm{H}^*\, N)^q.$$

This spectral sequence is of "homological type", and therefore converges without any additional assumptions.

Given a homotopy pullback square

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d] \\ Y' \arrow[r] & Y, \end{tikzcd}$$

we get an induced map

$$C^*(X) \otimes_{C^*(Y)} C^*(Y') \to C^*(X').$$

The conclusion of Theorem 6 is that this map induces an isomorphism on cohomology, so we have a spectral sequence with E_2-term

$$E_2^{p,q} = \mathrm{Tor}_{-p}^{\mathrm{H}^* Y}(\mathrm{H}^*\, X, \mathrm{H}^*\, Y')^q$$

converging to $\mathrm{H}^*(X')$. This is the classical *cohomological Eilenberg-Moore spectral sequence.*