Chapter 2

The homotopy theory of CW complexes

10 Serre fibrations and relative lifting

Relative CW complexes

We will do many proofs by induction over cells in a CW complex. We might as well base the induction arbitrarily. This suggests the following definition.

Definition 10.1. A *relative CW-complex* is a pair (X, A) together with a filtration

$$A = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \cdots \subseteq X,$$

such that (1) for all n the space X_n sits in a pushout square:

$$\begin{tikzcd} \coprod_{\alpha\in\Sigma_n} S^{n-1}_\alpha \arrow[r] \arrow[d] & \coprod_{\alpha\in\Sigma_n} D^n_\alpha \arrow[d] \\ X_{n-1} \arrow[r] & X_n \,, \end{tikzcd}$$

and (2) $X = \varinjlim X_n$ topologically.

The maps $S^{n-1} \to X_{n-1}$ are "attaching maps" and the maps $D^n \to X_n$ are "characteristic maps."

If $A = \varnothing$, this is just the definition of a CW-complex. Often X will be a CW-complex and A a subcomplex.

Serre fibrations

If we're going to restrict our attenton to CW complexes, we might as well weaken the lifting condition defining fibrations.

Definition 10.2. A map $p : E \to B$ is a *Serre fibration* if it has the homotopy lifting property ("HLP") with respect to all CW complexes. That is, for every CW complex X and every solid arrow diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\mathrm{in}_0"] & E \arrow[d, "p"] \\ X \times I \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

there is a lift as indicated.

For contrast, what we called a fibration is also known as a *Hurewicz fibration.* (Witold Hurewicz was a faculty member at MIT from 1945 till his death in 1958 from a fall from the top of the Uxmal Pyramid in Mexico.)

Clearly things like the homotopy long exact sequence of a fibration extend to the context of Serre fibrations. So for example:

Lemma 10.3. *Suppose that $p : E \to B$ is both a Serre fibration and a weak equivalence. Then each fiber is weakly contractible; i.e. the map to $*$ is a weak equivalence.*

Proof. Since $\pi_0(E) \to \pi_0(B)$ is bijective, we may assume that both E and B are path connected. The long exact homotopy sequence shows that $\partial : \pi_1(B) \to \pi_0(F)$ is surjective with kernel given by the image of the surjection $\pi_1(E) \to \pi_1(B)$: so $\pi_0(F) = *$. Moving up the sequence then shows that all the higher homotopy groups of F are also trivial. □

No new ideas are required to prove the following two facts.

Proposition 10.4. *Let $p : E \to B$. The following are equivalent.*

1. *p is a Serre fibration.*
2. *p has HLP with respect to D^n for all $n \geq 0$.*
3. *p has relative HLP with respect to $S^{n-1} \hookrightarrow D^n$ for all $n \geq 0$.*
4. *p has relative HLP with respect to $A \hookrightarrow X$ for all relative CW complexes (X, A).*

Proposition 10.5 (Relative straightening)**.** *Assume that (X, A) is a relative CW complex and that $p : E \to B$ is a Serre fibration, and that the diagram*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "j"] & E \arrow[d, "p"] \\ X \arrow[r, "g"] & B \end{tikzcd}$$

commutes. If g is homotopic to a map g' still making the diagram commute and for which there is a filler, then there is a filler for g.

Proof of "Whitehead's little theorem"

We are moving towards a proof of this theorem of J.H.C. Whitehead.

Theorem 10.6. *Let $f : X \to Y$ be a weak equivalence and W any CW complex. The induced map $[W, X] \to [W, Y]$ is bijective.*

The key fact is this:

Proposition 10.7. *Suppose that $j : A \hookrightarrow X$ is a relative CW complex and $p : E \to B$ is both a Serre fibration and a weak equivalence. Then a filler exists in any diagram*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "j"] & E \arrow[d, "p"] \\ X \arrow[r] \arrow[ur, dashed] & B . \end{tikzcd}$$

In the language of Quillen's *Homotopical Algebra*, this says that j satisfies the left lifting property with respect to "acyclic" Serre fibrations, and acyclic Serre fibrations satisfy the right lifting property with respect to relative CW complex inclusions.

Proof (following [29]). The proof will of course go by induction. The inductive step is this: Assuming that $p : E \to B$ is a Serre fibration and a weak equivalence, any diagram

$$\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d, "j"] & E \arrow[d, "p"] \\ D^n \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

admits a filler.

First let's think about the special case in which $B = *$. This is true because for any path connected space X the evident surjection

$$\pi_n(X, *) \to [S^n, X]$$

is none other than the orbit projection associated to the action of $\pi_1(X, *)$ on $\pi_n(X, *)$. This fact is why I wanted to focus on this otherwise rather obscure action. You'll verify it for homework.

For the general case, we begin by using Lemma 10.5 replacing the map g by a homotopic map g' with properties that will let us construct a filler. To define g', let $\varphi : D^n \to D^n$ by

$$\varphi : v \mapsto \begin{cases} 0 & \text{if } |v| \leq 1/2 \\ (2|v| - 1)v & \text{if } |v| \geq 1/2 \,. \end{cases}$$

This map is homotopic to the identity (by a piecewise linear homotopy that fixes S^{n-1}), so $g' = g \circ \varphi \simeq g$.

The virtue of g' is that we can treat the two parts of D^n separately. The annulus $\{v \in D^n : |v| \geq 1/2\}$ is homeomorphic to $I \times S^{n-1}$, so a lifting exists on it since p is a Serre fibration. On the other hand g' is constant on the inner disk $D^n_{1/2}$, with value $g(0)$. We just constructed a lift on $S^{n-1}_{1/2}$, but it actually lands in the fiber of p over $g(0)$. We can fill in that map with a map $D^n_{1/2} \to p^{-1}(g(0))$ since the fiber is weakly contractible. □

Proof of Theorem 10.6. Begin by factoring $f : X \to Y$ as a homotopy equivalence followed by a fibration; so as a weak equivalence followed by a Serre fibration p. Weak equivalences satisfy "2 out of 3" (as you'll check for homework), so p is again a weak equivalence. Thus we may assume that f is a Serre fibration (as well as being a weak equivalence).

To see that the map is onto, apply Proposition 10.7 to

$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ W \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

To see that the map is one-to-one, apply Proposition 10.7 to

$$\begin{tikzcd} W \times \partial I \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ W \times I \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

□

This style of proof – using lifting conditions and factorizations – is very much in the spirit of Daniel Quillen's formalization of homotopy theory in his development of "model categories."

11 Connectivity and approximation

The language of connectivity

An analysis of the proof of "Whitehead's little theorem" shows that if the CW complex we are using as a source has dimension at most n, then we only needed to know that the map $X \to Y$ was an "n-equivalence" in the following sense.

Definition 11.1. Let n be a positive integer. A map $f : X \to Y$ is an *n-equivalence* provided that $f_* : \pi_0(X) \to \pi_0(Y)$ is an isomorphism, and for every choice of basepoint $a \in X$ the map $f_* : \pi_q(X, a) \to \pi_q(Y, f(a))$ is an isomorphism for $q < n$ and an epimorphism for $q = n$. It is a *0-equivalence* if $f_* : \pi_0(X) \to \pi_0(Y)$ is an epimorphism.

So a map is a weak equivalence if it is an n-equivalence for all n. We restate:

Theorem 11.2. *Let n be a nonnegative integer and W a CW complex. If $f : X \to Y$ is an n-equivalence then the map $f_* : [W, X] \to [W, Y]$ is bijective if $\dim W < n$ and surjective if $\dim W = n$.*

The odd edge condition in the definition of n-equivalence might be made more palatable by noticing that the long exact homotopy sequence shows that (for $n > 0$) f is an n-equivalence if and only if $\pi_0(X) \to \pi_0(Y)$ is bijective and for any $b \in Y$ the group $\pi_q(F(f, b))$ is trivial for $q < n$.

This suggests some further language.

Definition 11.3. Let n be a positive integer. A space X is *n-connected* if it is path connected and for any choice of basepoint a the set $\pi_q(X, a)$ is trivial for all $q \leq n$. A space X is *0-connected* if it is path connected.

So "1-connected" and "simply connected" are synonymous. The homotopy long exact sequence shows that for $n > 0$ a map $X \to Y$ is an n-equivalence if it is bijective on connected components and for every $b \in Y$ the homotopy fiber $F(f, b)$ is n-connected.

The language of connectivity extends to pairs:

Definition 11.4. Let n be a non-negative integer. A pair (X, A) is *n-connected* if $\pi_0(A) \to \pi_0(X)$ is surjective and for every basepoint $a \in A$ the set $\pi_q(X, A, a)$ is trivial for $q \leq n$.

That is, (X, A) is n-connected if the inclusion map $A \to X$ is an n-equivalence.

Skeletal approximation

Theorem 11.5 (The skeletal approximation theorem)**.** *Let (X, A) and (Y, B) be relative CW complexes. Any map $f : (X, A) \to (Y, B)$ is homotopic rel A to a* skeletal *map – a map sending X_n into Y_n for all n. Any homotopy between skeletal maps can be deformed rel A to one sending X_n into Y_{n+1} for all n.*

I will not give a proof of this theorem. You have to inductively push maps off of cells, using smooth or simplicial approximation techniques. I am following Norman Steenrod in calling such a map "skeletal" rather than the more common "cellular," since it is after all not required to send cells to cells. See for example [4, p. 208]

Corollary 11.6. *Any map $X \to Y$ of CW complexes is homotopic to a skeletal map, and any homotopy between skeletal maps can be deformed to one sending X_n to Y_{n+1}.*

For example, the n-sphere $I^n/\partial I^n$ has a CW structure in which $\mathrm{Sk}_{n-1}S^n = *$ and $\mathrm{Sk}_n S^n = S^n$. The characteristic map is given by a choice of homeomorphism $D^n \to I^n$. So if $q < n$, then any map $S^q \to S^n$ factors through the basepoint up to homotopy. This shows that

$$\pi_q(S^n) = 0 \,\text{for}\, q < n$$

– the n-sphere is $(n-1)$-connected. So also is any CW complex with one 0-cell and no other q-cells for $q < n$.

As a special case (one used in proving the theorem in fact):

Proposition 11.7. *Let (X, A) be a relative CW complex in which all the cells of X are in dimension greater than n. Then (X, A) is n-connected.*

For example (with $A = \varnothing$) $\pi_0(X_0) \to \pi_0(X)$ is surjective: every path component of X contains a vertex. And $\pi_1(X_1) \to \pi_1(X)$ is surjective: any path between vertices can be deformed onto the 1-skeleton. Moreover, any homotopy between paths in the 1-skeleton can be deformed to lie in the 2-skeleton; $\pi_1(X_2) \to \pi_1(X)$ is an isomorphism.

For $n > 0$, this is saying that for any choice of basepoint in X, $\pi_q(X, X_n)$ is trivial for $q \le n$.

CW approximation

Any space is weakly equivalent to a CW complex. In fact:

Theorem 11.8. *Any map $f : A \to Z$ admits a factorization as*

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[rr, bend right, "f"'] & X \arrow[r, "j"] & Z \end{tikzcd}$$

where i is a relative CW inclusion and j is a weak equivalence.

This is analogous to the factorization as a cofibration followed by a homotopy equivalence. This factorization is part of the "Quillen model structure" on spaces, while the earlier one is part of the "Strøm model structure." An important special case: $A = \varnothing$: so any space admits a weak equivalence from a CW complex.

Proof. Fix a space Y. To begin with, pick a point in each path component of Y not meeting A and adjoin to A a discrete set mapping to those points. This gives us a factorization $A \to X_0 \to Y$ in which X_0 is obtained from A by attaching 0-simplices and $X_0 \to Y$ is a 0-equivalence.

Next, for each pair of distinct components of A that map to the same component in Y pick points a, b in them and a path in Y from $f(a)$ to $f(b)$. These data determine a map to Y from the pushout

$$\begin{tikzcd} \coprod S^0 \arrow[r] \arrow[d] & X_0 \arrow[d] \\ \coprod D^1 \arrow[r] & X_1' \end{tikzcd}$$

that is bijective on π_0.

These constructions let us assume that both A and Y are path connected, and we do so henceforth. Pick a point in A to use as a basepoint, and use its image in Y as a basepoint there.

We want to add 1-cells to A to obtain a path-connected space X, along with an extension of f to a 1-equivalence $X \to Y$. This just means a surjection in π_1. So pick a subset of $\pi_1(Y)$ that together with $\mathrm{im}(\pi_1(A) \to \pi_1(Y))$ generate $\pi_1(Y)$, and pick a representative loop for each element of that set. This defines a map $X = A \vee \bigvee S^1 \to Y$ that is surjective on π_1.

Now suppose that $f : A \to Y$ is a 1-equivalence. We will adjoin 2-cells to A to produce a space X, together with an extension of f to a 2-equivalence.

As a convenience, we first factor f as $A \hookrightarrow Y' \to Y$ in which the first map is a closed cofibration and the second is a homotopy equivalence. This lets us assume that A is in fact a subspace of Y.

We want to adjoin 2-cells to produce an extension of f to a 2-equivalence $X \to Y$. The group $\pi_2(Y, A)$ measures the failure of f itself to be a 2-equivalence. It is a group with an action of $\pi_1(A)$. Pick generators of it as such, and for each pick a representative map

$$(D^2, S^1, *) \to (Y, A, *)$$

Together they determine a map to Y from the pushout in

$$\begin{tikzcd}
\coprod S^1 \arrow[r] \arrow[d] & A \arrow[d] \\
\coprod D^2 \arrow[r] & X
\end{tikzcd}$$

We want to see that $\pi_1(X) \to \pi_1(Y)$ is an isomorphism and $\pi_2(X) \to \pi_2(Y)$ is an epimorphism. The factorization $A \to X \to Y$ determines a map of homotopy long exact sequences of groups:

$$\begin{tikzcd}
\pi_2(A) \arrow[r] \arrow[d] & \pi_2(X) \arrow[r] \arrow[d] & \pi_2(X, A) \arrow[r, "\partial"] \arrow[d] & \pi_1(A) \arrow[r] \arrow[d, "="] & \pi_1(X) \arrow[r] \arrow[d] & * \\
\pi_2(A) \arrow[r] & \pi_2(Y) \arrow[r] & \pi_2(Y, A) \arrow[r, "\partial"] & \pi_1(A) \arrow[r] & \pi_1(Y) \arrow[r] & *
\end{tikzcd}$$

By construction, the middle arrow is surjective. The usual diagram chases show that $\pi_1(X) \to \pi_1(Y)$ is an isomorphism and that $\pi_2(X) \to \pi_2(Y)$ is an epimorphism.

An identical argument continues the induction. We carried out this case because it's slightly nonstandard, involving nonabelian groups.

At the end, we have to observe that the direct limit of a sequence of cell attachments enjoys the property that

$$\lim_{\to} \pi_q(X_n) \to \pi_q(\lim_{\to} X_n)$$

is an isomorphism. □

Notice that if we only want to get to an n-equivalence, we need only add cells up to dimension n: Any space is n-equivalent to a CW complex of dimension at most n.

This construction is of course very ineffective: at each stage you have to compute some relative homotopy group! And since finite complexes have infinitely much homotopy, it seems that this process might go on for ever even for very simple spaces. The cellular chain complex of a CW complex suggests that one might be able to do better. In fact you can, as long as your space is simply connected.

Theorem 11.9 (C.T.C. Wall: [44],[43]). *Let Y be a simply connected space such that $H_n(Y)$ is finitely generated for all n. Let β_n be the nth Betti number (the rank of $H_n(Y)$) and let τ_n be the nth torsion number (the number of finite cyclic summands in $H_n(Y)$). Then there is a CW complex with $(\beta_n + \tau_{n-1})$ n-cells for each n that admits a weak equivalence to Y.*

This is clearly optimal, since in order to produce a finite cyclic summand in the nth homology of a chain complex of finitely generated abelian groups you need generators in dimension n and $n+1$.

12 The Postnikov tower

Postnikov sections

The cell attaching method used in the proof of CW approximation has other applications.

Theorem 12.1. *For any space X and any nonnegative integer n, there is a map $X \to P_n(X)$ with the following properties.*
(1) For every basepoint $* \in X$, $\pi_q(X,*) \to \pi_q(P_n(X),*)$ is an isomorphism for $q \le n$.
(2) For every basepoint $* \in P_n(X)$, $\pi_q(P_n(X),*) = 0$ for $q > n$.
(3) $(P_n(X), X)$ is a relative CW complex with cells of dimension not less than $(n+2)$.

When $n = 0$, the space $P_0(X)$ is "weakly discrete"; a CW approximation to it is given by a map $\pi_0(X) \to P_0(X)$.

When X is path connected and $n = 1$, this is asserting the existence of a path connected space $P_1(X)$ with $\pi_1(P_1(X)) = \pi_1(X)$ and *no higher homotopy groups*, and a map $X \to P_1(X)$ inducing an isomorphism on π_1. Assuming $P_1(X)$ is nice enough to have a universal cover, its universal cover will be weakly contractible. Such a space is said to be "aspherical." Thus any group G is the fundamental group of an aspherical space, because it occurs as $\pi_1(X)$ for a suitable 2-dimensional CW complex: Express G in terms of generators and relations; form a wedge of circles indexed by the generators, and map in a wedge of circles according to the relations. By the van Kampen theorem, the cofiber of this map will have the desired fundamental group.

Proof. Work one connected component at a time. We'll progressively clean out the higher homotopy of the space X, constructing a sequence of spaces

$$X = (n) \to X(n+1) \to X(n+2) \to \cdots$$

all sharing the same π_q for $q \le n$ but with

$$\pi_q(X(t)) = 0 \quad \text{for } n < q \le t\,.$$

We can take $X(n) = X$. Thereafter $X(t)$ will be built from $X(t-1)$ by attaching $(t+1)$-cells, so by Corollary 11.7 the pair $(X(t), X(t-1))$ is t-connected: the inclusion induces isomorphisms in π_q for $q < t$ and $\pi_t(X(t), X(t-1)) = 0$.

So we just want to be sure to kill $\pi_t(X(t-1))$, while not introducing anything new in $\pi_t(X(t))$. Pick a set of generators for $\pi_t(X(t-1))$, and pick representatives $S^t \to X(t-1)$ for them. Attach $(t+1)$-cells to $X(t-1)$ using these maps as attaching maps, to form a space $X(t)$. Here's a fragment of the homotopy long exact sequence.

$$\pi_{t+1}(X(t), X(t-1)) \xrightarrow{\partial} \pi_t(X(t-1)) \to \pi_t(X(t)) \to \pi_t(X(t), X(t-1)) = 0\,.$$

By construction, the boundary map is surjective, so $\pi_t(X(t)) = 0$.

Now pass to the limit;

$$P_n(X) = \lim_{\to} X(t)\,. \qquad \square$$

If X was a CW complex, we can use skeletal approximation to make all the attaching maps skeletal. They then join any cells of the same dimension in X, and the resulting space $P_n(X)$ admits the structure of a CW complex in which X is a subcomplex.

What's this about passing to the limit?

Lemma 12.2. *Any compact subspace of a CW complex lies in a finite subcomplex.*

Proof. The "interior" of D^n is $D^n\backslash S^{n-1}$ (so for example the interior of D^0 is D^0 itself). A CW complex X is, as a set, the disjoint union of the interiors of its cells. These subspaces are sometimes called "open cells," but since they are rarely open in X I prefer "cell interiors." Any subset of X that meets each cell interior in a finite set is a discrete subspace of X. So any compact subset of X meets only finitely many cell interiors. In particular a CW complex is compact if and only if it is finite.

The boundary of an n-cell (i.e. the image of the corresponding attaching map) is a compact subspace of the $(n-1)$-skeleton. It meets only finitely many of the cell interiors in that $(n-1)$-dimensional CW complex. By induction on dimension, all of those cells lie in finite complexes, so the n-cell we began with lies in a finite subcomplex.

Now let K be a compact subspace of X. It lies in the union of the finite subcomplexes containing the finite number of cell interiors meeting K. This union is a finite subcomplex of X. $\square$

If (X, A) is a relative CW complex, the quotient X/A is a CW complex, where we can apply this lemma.

Corollary 12.3. *Let $X(0) \subseteq X(1) \subseteq \cdots$ be a sequence of relative CW inclusions. Then for each q*

$$\lim_{\to} \pi_q(X(n)) \xrightarrow{\cong} \pi_q(\lim_{\to} X(n))$$

Proof. Both S^q and D^{q+1} are compact. $\square$

Now we have really gotten into homotopy theory! The space $P_n(X)$ is called the *nth Postnikov section* of X. (Mikhail Postnikov (1927–2004) worked at Steklov Institute in Moscow. This work was published in 1951.) Most of the time they are infinite dimensional, and you usually can't even compute their cohomology.

The Postnikov tower

How unique is the map $X \to P_n(X)$? How natural is this construction? To answer these questions, observe:

Proposition 12.4. *Let n be a nonnegative integer, and let Y be a space such that $\pi_q(Y, *) = 0$ for every choice of basepoint and all $q > n$. Let (X, A) be a relative CW complex. If all the cells in $X\backslash A$ are of dimension at least $n + 2$ then the map*

$$[X, Y] \to [A, Y]\,.$$

is bijective. If there are also $(n + 1)$-cells, the map is still injective.

Proof. This uses the fact that if $\pi_q(Y, *) = 0$ then any map $S^q \to Y$ landing in the path component containing $*$ extends to a map from D^{q+1}.

Surjectivity: We extend a map $A \to Y$ to a map from X. For each attaching map $g : S^{q-1} \to \mathrm{Sk}_{q-1}X$ (where $q \geq n+2$) the composite $f \circ g : S^{q-1} \to Y$ extends over the disk D^q since $q-1 > n$.

Injectivity: Regard $(X \times I, X \times \partial I \cup A \times I)$ as a relative CW complex, in which the cells are of dimension one larger than those of X. □

Corollary 12.5. *Let X be an n-connected CW complex and Y a space with homotopy concentrated in dimension at most n. Then every map from X to Y is homotopic to a constant map.*

Proof. By CW approximation, we may assume that X has a 0-cell and no other cells of dimension less than $n+1$. The pair $(X, *)$ satisfies the requirement necessary to conclude that $[X, Y] \to [*, Y]$ is injective. □

Now let $f : X \to Y$ be any map. Construct $X \to P_m(X)$ and $Y \to P_n(Y)$, so that $P_m(X)$ is attached using cells of dimension at least $m+2$ and $\pi_q(P_n(Y)) = 0$ for $q > n$. If $m \geq n$, then by Proposition 12.4 there is a unique homotopy class of maps $P_m(X) \to P_n(X)$ making

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ P_m(X) \arrow[r, dashed] & P_n(Y) \end{tikzcd}$$

commute.

For example we could take $X = Y$ and use the identity map: For $m \geq n$ there is a unique homotopy class $P_m(X) \to P_n(X)$ making

$$\begin{tikzcd} & X \arrow[dl] \arrow[dr] & \\ P_m(X) \arrow[rr, dashed] & & P_n(X) \end{tikzcd}$$

commute. When $m = n$, this shows that the map $X \to P_m(X)$ is unique up to a unique weak equivalence. When $m = n+1$, it gives us a tower of spaces, the *Postnikov tower*:

$$\begin{tikzcd} & \vdots \arrow[d] \\ & P_2(X) \arrow[d] \\ & P_1(X) \arrow[d] \\ X \arrow[r] \arrow[ur] \arrow[uur] & P_0(X) . \end{tikzcd}$$

As you go up in the tower you capture more and more of the homotopy groups of X. The Postnikov tower is functorial on the level of the homotopy category. We have a lot of control over how each space $P_n(X)$ is constructed, but very little control over what the resulting space looks like – e.g. what its homology is in high dimensions. There is likely to be a lot, even if X is a finite complex.

In a weak sense this tower is Eckmann-Hilton dual to a skeleton filtration: instead of building up a space as a direct limit of a sequence of spaces approximating the homology dimension by dimension, we are building it as the inverse limit of a sequence approximating the homotopy dimension by dimension.

More generally, Proposition 12.4 shows that $X \to P_n(X)$ is the *initial* map (in Ho**Top**) to a space with nontrivial homotopy only in dimension at most n.

Another common notation for $P_n(X)$ is $\tau_{\leq n}X$: the "truncation" of X at dimension n.

13 Hurewicz, Moore, Eilenberg, Mac Lane, and Whitehead

Hurewicz theorem

I have claimed that homotopy groups carry a lot of geometric information, but are correspondingly hard to compute. Homology groups are much easier; they are "local," in the sense that you can compute the homology of pieces of a space and glue the results together using Mayer-Vietoris. A cell structure quickly determines the homology (as we'll recall in the next lecture).

So it would be great if we had a way to compare homotopy and homology, maybe by means of a map

$$h : \pi_n(X) \to H_n(X) .$$

First we have to fix an orientation for the sphere $S^n = I^n/\partial I^n$ (for $n > 0$). Do this by declaring the standard ordered basis to be positively ordered. This gives us a preferred generator $\sigma_n \in H_n(S^n)$.

Now let $\alpha \in \pi_n(X)$. This homotopy class of maps $S^n \to X$ determines a map $H_n(S^n) \to H_n(X)$. Define

$$h(\alpha) = \alpha_*(\sigma_n) .$$

This is a well-defined map $h : \pi_n(X) \to H_n(X)$, the *Hurewicz map.*

Lemma 13.1. *h is a homomorphism.*

Proof. The product in $\pi_n(X)$ is given by the composite

$$\begin{tikzcd} S^n \arrow[r, "\alpha\beta"] \arrow[d, "\delta"] & X \\ S^n \vee S^n \arrow[r, "\alpha\vee\beta"] & X \vee X \arrow[u, "\nabla"'] \end{tikzcd}$$

where δ pinches an equator and ∇ is the fold map. Apply $\overline{H}_n$ and trace where σ_n goes:

$$\begin{tikzcd} \sigma_n \arrow[d, mapsto] & h(\alpha)+h(\beta) \\ (\sigma_n,\sigma_n) \arrow[r, mapsto] & (h(\alpha),h(\beta)) \arrow[u, mapsto] \end{tikzcd} .$$

□

When $n = 1$, the Hurewicz homomorphism factors through the abelianization of $\pi_1(X)$.

Theorem 13.2 (Hurewicz). *If X is path-connected, $\pi_1(X)^{ab} \to H_1(X)$ is an isomorphism. If X is $(n-1)$-connected for $n > 1$, $\pi_n(X) \to H_n(X)$ is an isomorphism.*

This can be proved by "elementary means," but we'll prove an improved form of this theorem later and I'd prefer to defer the proof. The $n = 1$ case is due to Poincaré.

This lowest dimension in which homotopy can occur is the "Hurewicz dimension." If X is an $(n-1)$-connected CW complex, it has a CW approximation that begins in dimension n, and the reduced homology (being isomorphic to the cellular homology) vanishes below dimension n.

In the simply connected case there is a converse.

Corollary 13.3. *Let X be a simply connected space. If $\overline{H}_q(X) = 0$ for $q < n$ then X is $(n-1)$-connected.*

Proof. If $n > 2$, the Hurewicz theorem says that $\pi_2(X) = H_2(X) = 0$, so X is 2-connected. And so on. □

Simple connectivity is required here. A good example is provided by the "Poincaré sphere." Let I be the group of orientation-preserving symmetries of the regular icosohedron. It is a subgroup of $SO(3)$ of order 60. Its preimage $\widetilde{I}$ in the double cover S^3 of $SO(3)$ is a perfect group (of order 120). The quotient space $S^3/\widetilde{I}$ thus has $H_1 = 0$, and so by Poincaré duality $H_2 = 0$ as well. The group acts freely by oriented diffeomorphisms, so the quotient is an oriented 3-manifold with the same homology as S^3. But its fundamental group is $\tilde{I}$, so it is not even homotopy equivalent to S^3 . . . and it's certainly not 2-connected. You can't decide whether or not you need 1-cells or 2-cells by looking at homology alone, in this non-simply connected example. In fact $\widetilde{I}$ can be presented with two generator and two relations, so $S^3/\widetilde{I}$ has a CW structure with two 1-cells and two 2-cells. The boundary map $C_2 \to C_1$ is an isomorphism.

Moore spaces

A *Moore space* is a simple space with only one nonzero reduced homology group.

Proposition 13.4. *Let π be an abelian group and n a positive integer. There is a CW complex M with cells in dimensions 0, n, and $n+1$, such that*

$$\overline{H}_q(M) = \begin{cases} \pi & \text{if } q = n \\ 0 & \text{otherwise}. \end{cases}$$

Proof. If π is a free abelian group, we can pick generators and take a corresponding wedge of n-spheres.

For a general abelian group π, pick a resolution by free abelian groups,

$$0 \leftarrow \pi \leftarrow F_0 \xleftarrow{d} F_1 \leftarrow 0\,.$$

Pick generators for F_0 and F_1, say $\{\alpha_i : i \in I\}$ and $\{\beta_j : j \in J\}$. Build the corresponding wedges of n-spheres. If we can realize the map d as $\overline{H}_n(f)$ for some map between those wedges, then we can take M to be the mapping cone.

A pointed map from a wedge is given by pointed maps from each factor. The map d is determined by

$$d\beta_j = \sum_i a_{ji}\alpha_i$$

for some set of integers $\{a_{ji}\}$, finitely nonzero for fixed j. For each i we have an inclusion

$$\mathrm{in}_i : S^n \to \bigvee_{i \in I} S^n$$

determining an element $\mathrm{in}_i \in \pi_n(\bigvee_i S^n)$. The sum

$$\sum_i a_{ji}\mathrm{in}_i \,.$$

determines a map from S^n to $\bigvee_i S^n$. Use this on the jth copy of $\bigvee_j S^n$ to get a map

$$\bigvee_i S^n \leftarrow \bigvee_j S^n$$

that realizes d. We can then build M as an $n+1$-dimensional CW complex by taking the mapping cone of this map. □

For example the Moore space for $\pi = \mathbb{Z}/2\mathbb{Z}$ and $n = 1$ is the familiar space $\mathbb{R}P^2$, and when $n > 1$ we can use $\Sigma^{n-1}\mathbb{R}P^2$.

By wedging together Moore spaces we can form a space with any prescribed sequence of homology groups.

Eilenberg Mac Lane spaces

Now let M be a Moore space for π, n. Our construction of it began with n-cells, so by skeletal approximation it has no homotopy below dimension n. (We don't need to appeal to Corollary 13.3 for this.) It probably has lots above dimension n, but we can kill all that by forming the Postnikov stage or truncation

$$P_n(M) = \tau_{\leq n} M$$

This is now a space with just one homotopy group, in dimension n. The Hurewicz theorem tells us that this single homotopy group is canonically isomorphic to π.

If $n = 1$ we can start with any group π, abelian or not, form the 2-dimensional complex we just made with $\pi_1 = \pi$, and form its Postnikov 1-section.

So we have now constructed a space with a single nonzero homotopy group, in dimension n. This is an *Eilenberg Mac Lane space*, denoted

$$K(\pi, n)\,.$$

You know some examples of Eilenberg Mac Lane spaces already.

- $K(\mathbb{Z}, 1) = S^1$. $K(\mathbb{Z}^n, 1) = (S^1)^n$.

- Any closed surface other than S^2 and $\mathbb{R}P^2$ has contractible universal cover and so is aspherical. There are many other examples of aspherical compact manifolds. But as soon as there is torsion in a group, the Eilenberg Mac Lane space is infinite dimensional.

- The space $\mathbb{R}P^n$ has S^n as universal cover, and as $n \to \infty$ the space S^n loses all its homotopy groups. So
$$K(\mathbb{Z}/2\mathbb{Z}, 1) = \mathbb{R}P^\infty\,.$$
Similarly,
$$K(\mathbb{Z}, 2) = \mathbb{C}P^\infty\,.$$

The Eilenberg Mac Lane space $K(\pi, n)$ can be constructed functorially in π. This is not the case with the Moore space construction. This is why I resisted incorporating the pair (π, n) into a symbol for a Moore space.

Sammy Eilenberg (1913–1998) was born in Poland and worked mainly at Columbia. In addition to constructing their spaces, he and Saunders Mac Lane (1909–2005, Chicago) wrote the foundational paper on category theory. Eilenberg wrote several foundational texts: *Homological Algebra* with Henri Cartan (1904–2008, Paris), and *Foundations of Algebraic Topology* with Norman Steenrod (1910–1971, Princeton University)

The Whitehead tower

One further thing we can do at this point: Endow X with a basepoint $*$ and form the homotopy fiber of the map $X \to \tau_{\leq n} X$. By the homotopy long exact sequence, the map from the homotopy fiber will induce isomorphisms in π_q for $q > n$, while the homotopy groups of the homotopy fiber will be trivial for $q \leq n$: it is n-connected. Let's write $\tau_{>n} X$ for this space. For example, $\tau_{>0} X$ is the basepoint component of X (assuming $X \to \pi_0(X)$ is continuous). $\tau_{\geq 2} X$ is the universal cover of X (assuming that X is path connected and is nice enough to admit a universal cover).

The example of covering spaces shows that $\tau_{>n} X \to X$ is not unique in quite the same sense that $X \to \tau_{>n} X$ is; you need a basepoint condition. In the pointed homotopy category, $\tau_{>n} X \to X$ is the terminal map from an n-connected space.

These spaces fit into a tower also, this time with X at the bottom:

$$\begin{tikzcd}
\vdots \arrow[d] & \\
\tau_{\geq 2} X \arrow[d] \arrow[ddr] & \\
\tau_{\geq 1} X \arrow[d] \arrow[dr] & \\
\tau_{\geq 0} X \arrow[r, "="] & X
\end{tikzcd}$$

This is the *Whitehead tower*. (George Whitehead, 1918–2004, MIT faculty member, was apparently related neither to Alfred North Whitehead nor to J.H.C. Whitehead. John Moore (1923–2016, working at Princeton) was a student of his, by the way (and an MIT alum), and I was a student of Moore's.)

14 Representability of cohomology

I want to think a little more about the significance of Eilenberg Mac Lane spaces. First, how unique are they?

Let π be an abelian group and n a positive integer. Pick a free resolution

$$0 \to F_1 \to F_0 \to \pi \to 0\,,$$

pick generators for F_0 and F_1, and build the corresponding cofiber sequence

$$\bigvee_j S^n \to \bigvee_i S^n \to M\,.$$

So M is a Moore space with $H_n(M) = \pi$. Our first model for $K(\pi, n)$ is the Postnikov section $\tau_{\leq n} M$.

Lemma 14.1. *Let n be a positive integer and let Y be any pointed space such that $\pi_q(Y, *) = 0$ for $q \neq n$, and write G for $\pi_n(Y, *)$. Then*

$$\pi_n : [\tau_{\leq n} M, Y]_* \to \mathrm{Hom}(\pi, G)$$

is an isomorphism.

Proof. Since $M \to \tau_{\leq n} M$ is universal among maps to spaces with homotopy concentrated in dimensions at most n, it's enough to show that

$$\pi_n : [M, Y]_* \to \mathrm{Hom}(\pi, G)$$

is an isomorphism. Since the sequence defining M is co-exact, we have an exact sequence

$$[\bigvee_j S^n, Y]_* \leftarrow [\bigvee_i S^n, Y]_* \leftarrow [M, Y]_* \leftarrow [\bigvee_j S^{n+1}, Y]_* \,.$$

Our assumptions on Y imply that this sequence reads

$$\mathrm{Hom}(F_1, G) \leftarrow \mathrm{Hom}(F_0, G) \leftarrow [M, Y]_* \leftarrow 0 \,.$$

But a homomorphism $F_0 \to G$ that restricts to zero on F_1 is exactly a homomorphism $\pi \to G$. $\square$

We phrased this for π and G abelian, but if $n = 1$ the same proof works with both groups arbitrary.

In particular, we could take $G = \pi$, and discover that there is a unique homotopy class of maps $\tau_{\leq n} M \to Y$ inducing the identity in π_n. This map is a weak equivalence. So if Y is also a CW complex, the map is a homotopy equivalence.

We learn from this that any two CW complexes of type $K(\pi, n)$ are homotopy equivalent by a homotopy equivalence inducing the identity on π_n, and that that homotopy equivalence is unique up to homotopy. This leads to:

Corollary 14.2. *For any positive integer n there is a functor*

$$\mathbf{Ab} \to \mathrm{Ho}(\mathbf{CW}_*)$$

sending π to a space of type $K(\pi, n)$, unique up to isomorphism. When $n = 1$ this extends to a functor

$$\mathbf{Gp} \to \mathrm{Ho}(\mathbf{CW}_*) \,.$$

In fact it is possible to construct $K(\pi, n)$ as a functor from $\mathbf{Ab}$ to the category of topological abelian groups.

The case $n = 1$ is due to Heinz Hopf: There is, up to homotopy, a unique aspherical space with any prescribed fundamental group. The theory of covering spaces can be used in that case to check functoriality. This provides a collection of invariants of groups, $H_n(K(\pi, 1); G)$ and $H^n(K(\pi, 1); G)$. More generally, any π-module M determines a local coefficient system $\widetilde{M}$ over $K(\pi, 1)$, and one then has local homology and cohomology groups. It's not hard to show these are the homology and cohomology of the group with these coefficients:

$$H_n(K(\pi, 1); \widetilde{M}) = \mathrm{Tor}_n^{\mathbb{Z}[\pi]}(\mathbb{Z}, M) \,, \quad H^n(K(\pi, 1); \widetilde{M}) = \mathrm{Ext}^n_{\mathbb{Z}[\pi]}(\mathbb{Z}, M) \,.$$

Fundamental classes

Let n be a positive integer and Y an $(n-1)$-connected space. Then $\overline{H}_q(Y) = 0$ for $q < n$. Let π be an abelian group. The universal coefficient theorem asserts the existence of a short exact sequence

$$0 \to \mathrm{Ext}^1(H_{q-1}(Y), \pi) \to H^q(Y; \pi) \to \mathrm{Hom}(H_q(Y), \pi) \to 0$$

for any q. This shows that $H^q(Y; \pi) = 0$ for $q < n$. When $q = n$, the Ext term vanishes so the second map is an isomorphism. If we take $\pi = \pi_n(Y)$, for example, the inverse of the Hurewicz isomorphism is an element in Hom, and so delivers to us a canonical cohomology class in $H^n(Y; \pi_n(Y))$.

In particular, with $Y = K(\pi, n)$ we obtain a canonical class

$$\iota_n \in H^n(K(\pi, n); \pi)$$

called the *fundamental class*. Using it, we get a canonical natural transformation

$$[X, K(\pi, n)] \to H^n(X; \pi)$$

sending f to $f^*(\iota_n)$.

Theorem 14.3. *If X is a CW complex, this map is an isomorphism.*

That is: On CW complexes, cohomology is a *representable functor*; the representing object is the appropriate Eilenberg Mac Lane space; and ι_n is the universal n-dimensional cohomology class with coefficients in π.

Test cases: We decided that $K(\mathbb{Z}/2\mathbb{Z}, 1) = \mathbb{R}P^\infty$. So the claim is that $H^1(X; \mathbb{Z}/2\mathbb{Z}) = [X, \mathbb{R}P^\infty]$. We'll discuss this in more detail later, but $\mathbb{R}P^\infty$ carries the universal real line bundle, so the set of homotopy classes of maps into it (from a CW complex X) is in bijection with the set of isomorphism classes of real line bundles over X. As you may know, that set is indeed given by $H^1(X; \mathbb{Z}/2\mathbb{Z}) = \mathrm{map}(\pi_1(X), \mathbb{Z}/2\mathbb{Z})$.

Similar story for $H^2(X; \mathbb{Z}) = [X, \mathbb{C}P^\infty]$.

One other case is of interest:

$$H^1(X, \mathbb{Z}) = [X, S^1] .$$

Other cases are less geometric!

Proof of Theorem 14.3. We'll prove a pointed version of the statement:

$$[X, K(\pi, n)]_* \xrightarrow{\cong} \overline{H}^n(X; \pi) .$$

Fix π, and pick any sequence of Eilenberg Mac Lane CW complexes, $K(\pi, n)$, $n \geq 0$. Thus for example $K(\pi, 0)$ is a CW complex that is homotopy equivalent to the discrete group π: we can take it to *be* π as a discrete group if we want.

The space $\Omega K(\pi, n+1)$ accepts a map from $K(\pi, n)$ that is an isomorphism on π_n; a CW replacement for $\Omega K(\pi, n+1)$ thus serves as another model for $K(\pi, n)$. Thus $K(\pi, n)$ has the structure of an H-group. In fact one can use $\Omega^2 K(\pi, n+2)$, by the same argument; so this H-group structure is abelian, and the functor $[-, K(\pi, n)]_*$ takes values in abelian groups.

The map $[X, K(\pi, n)]_* \to \overline{H}^n(X; \pi)$ is a homomorphism. To see this, use the pinch map $\Sigma X \to \Sigma X \vee \Sigma X$ to produce a homomorphism

$$\overline{H}^{n+1}(\Sigma X; \pi) \times \overline{H}^{n+1}(\Sigma X; \pi) \to \overline{H}^{n+1}(\Sigma X \vee \Sigma X; \pi) \to \overline{H}^{n+1}(\Sigma X; \pi) .$$

The argument proving that π_2 is abelian shows that this map coincides with the addition in the group $\overline{H}^{n+1}(\Sigma X;\pi) = \overline{H}^n(X;\pi)$.

The group structure in

$$[X, K(\pi,n)]_* = [X, \Omega K(\pi, n+1)]_* = [\Sigma X, K(\pi, n+1)]_*$$

has the same source; so the map is a homomorphism by naturality.

Now I will try to prove that the map is an isomorphism by induction on skelata.

When $X = X_0$, we can agree that

$$\mathrm{map}_*(X_0,\pi) = \overline{H}^0(X_0,\pi)\,, \quad [X_0, K(\pi,n)]_* = 0 = \overline{H}^n(X_0;\pi)\,, \text{for } n > 0\,.$$

We may henceforth assume that X is connected. In general we have a cofiber sequence $\bigvee S^{q-1} \to X_{q-1} \to X_q$. It is co-exact and hence induces an exact sequence in $[-, K(\pi,n)]_*$. It also induces an exact sequence in reduced cohomology, one that can be regarded as coming from the same geometric source. Since both S^{q-1} and X_{q-1} are of dimension less than q, the map is an isomorphism for them. So by the 5-lemma it's an isomorphism on X_q.

There is still a limiting argument to worry about, if X is infinite dimensional. □

Remark 14.4. One can also prove directly that cohomology is a representable functor on CW complexes, and then define Eilenberg Mac Lane spaces as the representing objects. The relevant theorem is "Brown representability" [5]. (Edgar Brown is professor emeritus at Brandeis University.) The fact that contravariant functors satisfying the kind of "descent" embodied by the Mayer-Vietoris theorem are representable gives homotopy theory a special character. Most of the time you can just work with spaces, which are much more concrete than functors!

Remark 14.5. Note that the suspension isomorphism in reduced cohomology is represented by the weak equivalence

$$K(\pi,n) \to \Omega K(\pi, n+1)$$

adjoint to the map representing the suspension of the fundamental class. A family of pointed spaces $\ldots, E_0, E_1, \ldots$ equipped with maps $E_n \to \Omega E_{n+1}$ (or equivalently $\Sigma E_n \to E_{n+1}$) is a *(topological) spectrum*. It's an *Ω-spectrum* if the maps $E_n \to \Omega E_{n+1}$ are all weak equivalences. Much of what we just did above carries over to Ω-spectra in general; the (abelian!) groups

$$\overline{E}^n(X) := [X, E_n]_*$$

form the groups in a (reduced generalized) cohomology theory. There are many examples. Any generalized cohomology theory is representable on CW complexes by an Ω spectrum.

Remark 14.6. One asset of representability is the "Yoneda lemma": Given a functor $F : \mathcal{C} \to \mathbf{Set}$ and an object Y in $\mathcal{C}$, we get inverse isomorphisms

$$\begin{aligned} \mathrm{n.t.}(\mathcal{C}(-,Y), F) &\rightleftarrows F(Y) \\ \theta &\mapsto \theta_Y(1_Y) \\ (f \mapsto f^*(y)) &\leftarrow\!\shortmid y \end{aligned}$$

In particular

$$\mathrm{n.t.}(\mathcal{C}(-,Y), \mathcal{C}(-,Z)) = \mathcal{C}(Y,Z)\,.$$

So for example

$$\mathrm{n.t.}(H^m(-,A), H^n(-,B)) = [K(A,m), K(B,n)] = H^n(K(A,m);B)\,.$$

Understanding the natural transformations acting between different dimensions of $H^*(-;\mathbb{F}_2)$, for example, is addressing the optimal value category for mod 2 cohomology. It's a graded $\mathbb{F}_2$ algebra, yes, but much more as well. This is the story of Steenrod operations, and it's addressed in full by computing $H^*(K(\mathbb{F}_2,n);\mathbb{F}_2)$.

15 Obstruction theory

Cellular homology

Let (X,A) be a relative CW-complex with skelata

$$A = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \cdots \subseteq X\,.$$

The inclusion $X_{n-1} \hookrightarrow X_n$ is a cofibration, so $H_*(X_n,X_{n-1}) \cong \overline{H}_*(X_n/X_{n-1})$. A choice of cell structure establishes a homeomorphism

$$X_n/X_{n-1} = \bigvee_{i\in\Sigma_n} S^n_i\,,$$

where Σ_n is the set of n-cells, so

$$H_*(X_n,X_{n-1}) \cong \mathbf{Z}[\Sigma_n]$$

This group is the *cellular chain group* $C_n = C_n(X,A)$.

There is a boundary map $d : C_{n+1} \to C_n$, defined by

$$d : C_{n+1} = H_{n+1}(X_{n+1},X_n) \xrightarrow{\partial} H_n(X_n) \to H_n(X_n,X_{n-1}) = C_n\,.$$

This gives us the *cellular chain complex*. In terms of the basis given by a choice of cell structure, the differential $d : C_{n+1} \to C_n$ is giving exactly the data of the *relative attaching maps*

$$S^n \xrightarrow{\alpha_i} X_n \to X_n/X_{n-1}$$

where α_i runs through the attaching maps of the $(n+1)$-cells. Passage to the relative attaching maps forgets a great deal of information about the homotopy type of X; homology is a rather weak invariant in this sense.

A theorem proved last term (at least when $A = \varnothing$) asserts that

$$H_n(X,A) \cong H_n(C_*(X,A))\,.$$

Of course, the same story runs for cohomology: one gets a chain complex which, in dimension n, is given by

$$C^n(X,A;\pi) = \mathrm{Hom}(C_n(X,A),\pi) = \mathrm{Map}(\Sigma_n,\pi)\,,$$

where π is any abelian group, and

$$H^n(X,A;\pi) = H^n(C^*(X,A;\pi)\,.$$

Obstruction theory

We've seen that when the dimension of the CW complex X is less than the connectivity of the space Y, any map from X to Y is null-homotopic. What if there is some overlap? Here's a more general type of question we can try to answer.

Question 15.1. Let $f : A \to Y$ be a map from a space A to Y. Suppose (X, A) is a relative CW-complex. When can we find an extension in the diagram below?

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, hook] & Y \\ X \arrow[ur, dashed] & \end{tikzcd}$$

We've seen that answering this kind of question can also lead to results about the uniqueness of an extension, by considering $X \times \partial I \cup A \times I \subseteq X \times I$.

Let's try to make this extension skeleton by skeleton, and find what obstructions occur. We can start easily enough! If Y is empty then A is too, and there's an extension if and only if X is empty as well.

More realistically, as long as Y is nonempty we can certainly extend to X_0 by sending the new points anywhere you like in Y.

So make such a choice: $f : X_0 \to Y$. Can we extend f further over X_1? Well, we can extend if and only if for every pair a and b of 0-cells in X_0 that are in the same path component of X_1, the images $f(a)$ and $f(b)$ are in the same path component in Y. Note that we might do better at this stage if we could go back and choose f better. This simple observation serves as a model for the whole process.

Let's now assume we have constructed $f : X_n \to Y$, for $n \geq 1$, and hope to extend it over X_{n+1}. Pick attaching maps for the $(n+1)$-cells, so we have the diagram

$$\begin{tikzcd} \coprod_{i\in\Sigma_{n+1}} S^n \arrow[r, "\alpha"] \arrow[d, hook] & X_n \arrow[r, "f"] \arrow[d] & Y \\ \coprod_{i\in\Sigma_{n+1}} D^n \arrow[r] & X_{n+1} \arrow[ur, dashed] & \end{tikzcd}$$

The desired extension exists if the composite $S^n \xrightarrow{\alpha_i} X_n \to Y$ is nullhomotopic for each $i \in \Sigma_{n+1}$.

Now is the moment to assume that Y is path connected and simple, so that

$$[S^n, Y] = \pi_n(Y, *)$$

canonically for any choice of basepoint. We will therefore omit basepoints from the notation.

This procedure produces a map $\theta_g : \Sigma_{n+1} \to \pi_n(Y)$, that is, an n-cochain, $\theta_f \in C^{n+1}(X, A; \pi_n(Y))$, and $\theta_f = 0$ if and only if f extends to a map $X_{n+1} \to Y$.

Proposition 15.2. *θ_f is a cocycle in $C^{n+1}(X, A; \pi_n(Y))$.*

Proof. θ_f gives a map $H_{n+1}(X_{n+1}, X_n) \to \pi_n(Y)$. We would like to show that the composite

$$H_{n+2}(X_{n+2}, X_{n+1}) \xrightarrow{\partial} H_{n+1}(X_{n+1}) \to H_{n+1}(X_{n+1}, X_n) \xrightarrow{\theta_f} \pi_n(Y)$$

is trivial.

We'll see this by relating the homotopy long exact sequence to the homology long exact sequence. A relative homotopy class is represented by a map

$$(I^q, \partial I^q, J_q) \to (X, A, *)\,.$$

Our choice of orientation for $I^q/\partial I^q$ specifies a generator for $H_q(I^q, \partial I^q)$. Evaluation of H_n then determines a map

$$h : \pi_q(X, A, *) \to H_q(X, A)\,,$$

the *relative Hurewicz homomorphism.* It is again a homomorphism, extending the definition of the absolute Hurewicz homomorphism, and gives us a map of long exact sequences.

The characteristic maps in the cell structure for X give us elements of $\pi_{n+1}(X_{n+1}, X_n)$ that map to the generators of $H_{n+1}(X_{n+1}, X_n)$.

These observations lead to part of the commutative diagram below.

$$\begin{tikzcd}
\pi_{n+2}(X_{n+2}, X_n) \arrow[r] \arrow[d, "\partial"] & H_{n+2}(X_{n+2}, X_{n+1}) \arrow[d, "\partial"] \arrow[dd, bend left=60, "d"] \\
\pi_{n+1}(X_{n+1}) \arrow[r] \arrow[d] \arrow[dd, bend right=60, "0"'] & H_{n+1}(X_{n+1}) \arrow[d] \\
\pi_{n+1}(X_{n+1}, X_n) \arrow[r] \arrow[d, "\partial"] & H_{n+1}(X_{n+1}, X_n) \arrow[d, "\theta_f"] \\
\pi_n(X_n) \arrow[r, "f_*"] & \pi_n(Y)
\end{tikzcd}$$

The bottom square commutes by definition of θ_f. Tracing around the left side goes through two successive maps in the homotopy long exact sequence, and so sends these elements to zero. □

This cochain θ_f is the "obstruction cocycle" associated to $f : X_n \to Y$. It obstructs the extension of f over the $(n+1)$-skeleton. This theorem gives a way of extending a map $A \to Y$ skeleton by skeleton all the way to a map $X \to Y$.

But it could happen that the extension you made to X_n doesn't admit a further extension to X_{n+1}, while some other extension to X_n would. In order to maintain some control, let's fix the extension to X_{n-1}, but allow the extension to X_n to vary.

Theorem 15.3. *Let (X, A) be a relative CW-complex and Y a path-connected simple space, and let $n \geq 1$. Let $f : X_n \to Y$ be a map from the n-skeleton of X, and let $\theta_f \in C^{n+1}(X, A; \pi_n(Y))$ be the associated obstruction cocycle. Then $f|_{X_{n-1}}$ extends to X_{n+1} if and only if $[\theta_f] \in H^{n+1}(X, A; \pi_n(Y))$ is zero.*

Proof. The proof begins with the construction a "difference cochain" δ associated to maps $f', f'' : X_n \to Y$ together with a homotopy from $f'|_{X_{n-1}}$ to $f''|_{X_{n-1}}$ rel A. It will not be a cocycle. Instead, it will provide a homology between the obstruction cocycles associated to f' and f''.

We'll lighten notation by dropping indication of the subspace A. Fix a cell structure on X. This is about homotopies, so let's begin by giving $X \times I$ the CW structure in which

$$(X \times I)_n = (X_n \times \partial I) \cup (X_{n-1} \times I)\,.$$

Each n-cell e in X produces in $X \times I$ an $(n+1)$-cell $e \times I$ and two n-cells $e \times 0$ and $e \times 1$. Thus there is a map

$$- \times I : C_n(X) \to C_{n+1}(X \times I)\,,$$

given by linearly extending the assignment on cells. This is not a chain map; rather

$$d(e \times I) = (de) \times I + (-1)^n (e \times 1 - e \times 0)$$

(by choice of orientation of the unit interval).

This construction defines a map

$$C^{n+1}(X; \pi_n(Y)) \to C^n(X; \pi_n(Y)))$$

by sending a cochain c to $e \mapsto c(e \times I)$.

Define a map $g : (X \times I)_n \to Y$ as follows. Send $X_n \times 0$ by f_0, $X_n \times 1$ by f_1, and $X_{n-1} \times I$ by a homotopy between the restrictions of f_0 and f_1 to X_{n-1}. We then have the obstruction cocycle $\theta_g \in C^{n+1}(X \times I; \pi_n(Y))$ associated to the map g.

Our difference cochain $\delta \in C^n(X; \pi_n(Y))$ is defined by

$$\delta(e) = \theta_g(e \times I) \, .$$

For any n-cell e in X, calculate as follows, using the definition of the differential in the cellular cochain complex:

$$0 = (d\theta_g)(e \times I) = \theta_g(d(e \times I)) = \theta_g((de) \times I) \pm (\theta_g(e \times 0) - \theta_g(e \times 1)) \, .$$

The three terms can be re-expressed as follows.

$$\theta_g((de) \times I) = \delta(de) = (d\delta)(e) \, ,$$

$$\theta_g(e \times 0) = \theta_{f'}(e) \, , \quad \theta_g(e \times 1) = \theta_{f''}(e) \, .$$

This verifies that

$$d\delta = \pm(\theta_{f'} - \theta_{f''}) \, .$$

So for a map $f : X_n \to Y$, the cohomology class of the obstruction cocycle θ_f depends only on $f|_{X_{n-1}}$. In particular if $f|_{X_{n-1}}$ does extend to a map from X_{n+1}, then this cohomology class vanishes.

For the converse, we observe that for any $f' : X_n \to Y$ and $\delta \in C^n(X; \pi_n(Y))$ there exists an extension f'' of $f'|_{X_{n-1}}$ such that δ is precisely the difference cochain associated to the pair (f', f'') and the constant homotopy between their restrictions to X_{n-1}. We leave this to you; it uses the homotopy extension property.

We can now argue as follows. Suppose that $[\theta_{f'}] = 0 \in H^{n+1}(X; \pi_n(Y))$. Pick a null-homology δ of $\theta_{f'}$, and pick f'' in such a way that δ is the difference cocycle between f' and f''. Then (adjusting the sign if necessary)

$$\theta_{f''} = \theta_{f'} - d\delta = 0 \, ,$$

so f'' extends to X_{n+1}. □

The easiest way to check that an obstruction class vanishes is to know that it lies in a zero group.

Corollary 15.4. *Let Y be a path connected simple space and (X, A) a relative CW complex. If $H^{n+1}(X, A; \pi_n(Y)) = 0$ for all $n \geq 1$ then any map $A \to Y$ extends to a map $X \to Y$. If moreover $H^n(X, A; \pi_n(Y)) = 0$ for all $n \geq 1$ then such an extension is unique up to homotopy rel A.*

Proof. The second assertion follows from the isomorphism

$$H^{n+1}(X \times I, A \times I \cup X \times \partial I; \pi) = H^n(X, A; \pi)\,. \qquad \square$$

This raises important questions. The reduced cohomology of a space may well be trivial with coefficients in a finite p-group, for a fixed prime p, for example. Are there homological conditions on Y guaranteeing that each homotopy group is a finite p-group? The power to prove results of that sort is part of the revolution in homotopy theory engineered by Jean-Pierre Serre, developments we will get to later in this course.

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