18.906: Problem Set II

Homework is an important part of this class. I hope you gain from the struggle. Collaboration can be effective, but be sure that you grapple with each problem on your own as well. If you do work with others, you must indicate with whom on your solution sheet.

Extra credit for finding mistakes and telling me about them early!

6. (a) Show that weak equivalences satisfy "2 out of 3": in

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "gf"'] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

if two of f, g, and gf are weak equivalences then so is the third.

(b) Let $f : \Sigma X \to Y$ be a pointed map, and let $\hat{f} : X \to \Omega Y$ be its adjoint. Construct a map $g : CX \to PY$ from the cone to the path space $PY = Y_*^I$ such that the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\hat{f}"] & CX \arrow[r] \arrow[d, "g"] & \Sigma X \arrow[d, "f"] \\ \Omega Y \arrow[r] & PY \arrow[r] & Y \end{tikzcd}$$

commutes.

7. Let $f : X \to Y$ and fix $* \in Y$. Assume that Y is path-connected. We've defined the homotopy fiber of f over $*$ to be the space

$$F(f, *) = \{(x, \omega) \in X \times Y^I : \omega(0) = *, \omega(1) = f(x)\}\,.$$

It comes equipped with a fibration $p : F(f, *) \to X$ sending (x, ω) to x. The loop space $\Omega(Y, *)$ "acts" on the homotopy fiber $F(f, *)$ from the right: let $\omega \in \Omega(Y, *)$ and $(x, \sigma) \in F(f, *)$, and define

$$(x, \sigma) \cdot \omega = (x, \sigma \cdot \omega)$$

where

$$(\sigma \cdot \omega)(t) = \begin{cases} \omega(2t) & 0 \le t \le 1/2 \\ \sigma(2t-1) & 1/2 \le t \le 1\,. \end{cases}$$

In particular, taking $X = *$, we get the usual "multiplication" $\Omega Y \times \Omega Y \to \Omega Y$, which is known to be associative and unital up to homotopy (and to admit a homotopy inverse, sending ω to $\overline{\omega} : t \mapsto \omega(1-t)$). The same proof shows that the action of $\Omega(Y, *)$ on $F(f, *)$ is associative and unital up to homotopy.

Suppose a group G acts on a set S (from the right) with orbit space X. The fiber product $S \times_X S$ consists of pairs of elements in the same orbit. The action is free exactly when the map $S \times G \to S \times_X S$, sending (s, g) to (s, sg), is bijective.

Returning to the story of the homotopy fiber, note that $p((x,\sigma)\cdot\omega) = x = p(x,\sigma)$. We get a map

$$F(f,*)\times\Omega(Y,*)\to F(f,*)\times_X F(f,*)$$

to the fiber product by sending $((x,\sigma),\omega)$ to $((x,\sigma),(x,\sigma)\cdot\omega)$.

Finally, the problem: Show that this map is a homotopy equivalence.

So in a sense there is a "free" (a better word would be "principal") action of $\Omega(Y,*)$ on $F(f,*)$ with orbit space X. In particular, taking f to be the identity map $X\to X$ $F(f,*)$ is the contractible path space PX; so X is the "orbit space" of an action of ΩX on a contractible space. This entitles us to regard X as the "classifying space" of ΩX.

8. By passing to π_0, the action described in **7.** provides a right action of the group $\pi_1(Y,*)$ on $\pi_0(F(f,*))$.

(a) Show that two elements in $\pi_0(F(f,*))$ map to the same element of $\pi_0(X)$ if and only if they are in the same orbit under this action.

(b) Suppose ω is a path in Y from $*$ to y. Write $\omega_\# : \pi_1(Y,*)\to\pi_1(Y,y)$ for the group isomorphism sending σ to $\omega\sigma\omega^{-1}$. Show that the isotropy group of the component of (x,ω) in $F(f,*)$ is

$$\omega_\#^{-1}\mathrm{im}\,(\pi_1(X,x)\to\pi_1(Y,f(x)))\subseteq\pi_1(Y,*)\,.$$

(c) Suppose that X is path connected, and pick $*\in X$. Conclude from **(a)** that the evident surjection $\pi_n(X,*)\to[S^n,X]$ can be identified with the orbit projection for the action of $\pi_1(X,*)$ on $\pi_n(X,*)$.

9. (a) Verify the "Peiffer identity," describing a relationship between the boundary map $\partial:\pi_2(X,A,*)\to\pi_1(A,*)$ and the action map $\cdot:\pi_1(A,*)\times\pi_2(X,A,*)\to\pi_2(X,A,*)$: For $\alpha,\beta\in\pi_2(X,A,*)$,

$$\alpha\beta\alpha^{-1} = (\partial\alpha)\cdot\beta\,.$$

This (along with the $\pi_1(A,*)$-equivariance of ∂, $\partial(\omega\cdot\alpha)=\omega(\partial\alpha)\omega^{-1}$) establishes this pair of groups as a "crossed module" – the first example, historically, due to J.H.C. Whitehead.

(b) Let $p:E\to B$ be a fibration, with fiber F over the nondegenerate basepoint $*$. Show that for any $n\geq 0$ and any basepoint $*\in F$, the projection map induces an isomorphism $\pi_n(E,F,*)\to\pi_n(B,*)$. (This is "dual" to the fact that if an inclusion $i:A\to X$ is a cofibration then $H_n(X,A)\to\overline{H}_n(X/A)$ is an isomorphism.)

This puts a condition on a map $f:F\to E$ that is necessary for it to be homotopic to the inclusion of the fiber in a fibration: $\pi_2(M(f),F,*)$ has to be abelian for any choice of basepoint in $*$.

(c) Verify the more general statement that if $A\subseteq B$ and $E_A=p^{-1}A$, then $\pi_*(E,E_A)\to\pi_*(B,A)$ is an isomorphism.

10. Suppose that the path connected space X has universal cover $\tilde{X}$. Pick a basepoint $\tilde{a}$ in $\tilde{X}$ and write a for its image in X. Show that for all $n\geq 2$ the projection map p induces an isomorphism $\pi_n(\tilde{X},\tilde{a})\to\pi_n(X,a)$.

Unique path lifting shows that a loop ω at a in X lifts to a unique path $\tilde{\omega}$ with $\tilde{\omega}(0) = \tilde{a}$. Write $\tilde{b}$ for $\tilde{\omega}(1)$; then $p(\tilde{b}) = a$ as well. The theory of covering spaces gives us a "deck transformation" (i.e. covering the identity map of X) $\omega_* : \tilde{X} \to \tilde{X}$ uniquely specified by requiring that $\omega_*(\tilde{a}) = \tilde{b}$.

The path ω induces an automorphism of $\pi_n(X, a)$, written $\omega_\#$. Use $\tilde{\omega}$ and ω_* to provide a map along the top of the commutative diagram

$$\begin{tikzcd}
\pi_n(\tilde{X}, \tilde{a}) \arrow[r] \arrow[d, "p_*"] & \pi_n(\tilde{X}, \tilde{a}) \arrow[d, "p_*"] \\
\pi_n(X, a) \arrow[r, "\omega_\#"] & \pi_n(X, a)
\end{tikzcd}$$

Visualize this in case $X = S^1 \vee S^2$ (taking $\alpha \in \pi_2(S^1 \vee S^2)$ and $\omega \in \pi_1(S^1 \vee S^2)$ to be the inclusions). What is $\pi_2(S^1 \vee S^2, *)$ as a $\mathbb{Z}[\pi_1(S^1 \vee S^2, *)]$-module? ($\pi_2(S^2) = \mathbb{Z}$, generated by the identity map.)