36 The fully relative cap product

Čech cohomology appeared as the natural algebra acting on $H^*(X, X-K)$, where K is a closed subspace of X:

$$\cap : \check{H}^p(K) \otimes H_n(X, X-K) \to H_q(X, X-K)\,, \quad p+q=n\,.$$

If we fix $x_K \in H_n(X, X-K)$, then capping with x_K gives a map

$$\cap x_K : \check{H}^p(K) \to H_q(X, X-K)\,, \quad p+q=n\,.$$

We will be very interested in showing that this map is an isomorphism under certain conditions. This is a kind of duality result, comparing cohomology and relative homology and reversing the dimensions. We'll try to show that such a map is an isomorphism by embedding it in a map of long exact sequences and using the five-lemma.

For a start, let's think about how these maps vary as we change K. So let L be a closed subset of K, so $X-K \subseteq X-L$ and we get a "restriction map"

$$i_* : H_n(X, X-K) \to H_n(X, X-L)\,.$$

Define x_L as the image of x_K. The diagram

$$\begin{array}{ccc}
\check{H}^p(K) & \longrightarrow & \check{H}^p(L) \\
\downarrow{\scriptstyle -\cap x_K} & & \downarrow{\scriptstyle -\cap x_L} \\
H_q(X, X-K) & \longrightarrow & H_q(X, X-L)
\end{array}$$

commutes by the projection formula. This embeds into a ladder shown in the theorem below. We will accompany this ladder with a second one, to complete the picture.

Theorem 36.1. *Let $L \subseteq K$ be closed subspaces of a space X. There is a "fully relative" cap product*

$$\cap : \check{H}^p(K, L) \otimes H_n(X, X-K) \to H_q(X-L, X-K)\,, \quad p+q=n\,,$$

such that for any $x_K \in H_n(X, X-K)$ the ladder

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & \check{H}^p(K,L) & \longrightarrow & \check{H}^p(K) & \longrightarrow & \check{H}^p(L) & \xrightarrow{\delta} & \check{H}^{p+1}(K,L) & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle \cap x_K} & & \downarrow{\scriptstyle \cap x_K} & & \downarrow{\scriptstyle \cap x_L} & & \downarrow{\scriptstyle \cap x_K} & & \\
\cdots & \longrightarrow & H_q(X-L, X-K) & \longrightarrow & H_q(X, X-K) & \longrightarrow & H_q(X, X-L) & \xrightarrow{\partial} & H_{q-1}(X-L, X-K) & \longrightarrow & \cdots
\end{array}$$

commutes, where x_L *is the restriction of* x_K *to* $H_n(X, X - L)$*, and for any* $x \in H_n(X)$

$$\begin{array}{ccccccccc} \cdots \to & \check{H}^p(X,K) & \to & \check{H}^p(X,L) & \to & \check{H}^p(K,L) & \xrightarrow{\delta} & \check{H}^{p+1}(X,K) & \to \cdots \\ & \downarrow{\scriptstyle \cap x} & & \downarrow{\scriptstyle \cap x} & & \downarrow{\scriptstyle \cap x_K} & & \downarrow{\scriptstyle \cap x} & \\ \cdots \to & H_q(X-K) & \to & H_q(X-L) & \to & H_q(X-L, X-K) & \xrightarrow{\partial} & H_{q-1}(X-K) & \to \cdots \end{array}$$

commutes, where x_K *is the restriction of* x *to* $H_n(X, X - K)$*.*

Proof. What I have to do is define a cap product along the bottom row of the diagram (with $p + q = n$)

$$\begin{array}{ccc} \check{H}^p(K) \otimes H_n(X, X-K) & \xrightarrow{\cap} & H_q(X, X-K) \\ \uparrow & & \uparrow \\ \check{H}^p(K,L) \otimes H_n(X, X-K) & \overset{\cap}{\dashrightarrow} & H_q(X-L, X-K) \end{array}$$

This requires going back to the origin of the cap product. Our map $\check{H}^p(K) \otimes H_n(X, X - K) \to H_q(X, X - K)$ came (via excision) from a chain map $S^p(U) \otimes S_n(U, U - K) \to S_q(U, U - K)$ where $U \supseteq K$, defined by $\beta \otimes \sigma \mapsto \beta(\sigma \circ \alpha_p) \cdot (\sigma \circ \omega_q)$. Now given inclusions

$$\begin{array}{ccc} L & \subseteq & K \\ \cap\!| & & \cap\!| \\ V & \subseteq & U \end{array}$$

we can certainly fill in the bottom row of the diagram

$$\begin{array}{ccc} S^p(U) \otimes S_n(U)/S_n(U-K) & \longrightarrow & S_q(U)/S_q(U-K) \\ \uparrow & & \uparrow \\ S^p(U,V) \otimes S_n(U)/S_n(U-K) & \longrightarrow & S_q(U-L)/S_q(U-K) \end{array}$$

Since cochains in $S^p(U, V)$ kill chains in V, we can extend the bottom row to

$$\begin{array}{ccc} S^p(U) \otimes S_n(U, U-K) & \longrightarrow & S_q(U, U-K) \\ \uparrow & & \uparrow \\ S^p(U,V) \otimes (S_n(U-L) + S_n(V))/S_n(U-K) & \longrightarrow & S_q(U-L)/S_q(U-K) \\ \downarrow{\scriptstyle \simeq} & & \\ S^p(U,V) \otimes S_n(U)/S_n(U-K) & & \end{array}$$

But $L \subseteq V$, so $(U-L) \cup V = U$, and the locality principle then guarantees that $S_n(U-L)+S_n(V) \to S_n(U)$ is a quasi-isomorphism. By excision, $H_n(U, U - K) \to H_n(X, X - K)$ is an isomorphism. Now use our standard map $\mu : H_*(C) \otimes H_*(D) \to H_*(C \otimes D)$.

This gives the construction of the fully relative cap product. We leave the checks of commutativity to the listener. □

The diagram

$$\begin{array}{ccc} \check{H}^p(L) & \xrightarrow{\delta} & \check{H}^{p+1}(K,L) \\ \downarrow{\scriptstyle -\cap x_L} & & \downarrow{\scriptstyle -\cap x_K} \\ H_q(X, X-L) & \xrightarrow{\partial} & H_{q-1}(X-L, X-K) \end{array}$$

provides us with the memorable formula

$$(\delta b) \cap x_K = \partial(b \cap x_L)\,.$$

The construction of the Mayer-Vietoris sequences now gives:

Theorem 36.2. *Let A, B be closed in a normal space or compact in a Hausdorff space. The Čech cohomology and singular homology Mayer-Vietoris sequences are compatible: for any $x_{A\cup B} \in H_n(X, X - A \cup B)$, there is a commutative ladder (where again we use the notation $H_q(X|A) = H_q(X, X - A)$, and again $p + q = n$)*

$$\begin{array}{ccccccccccc}
\cdots \longrightarrow & \check{H}^p(A\cup B) & \longrightarrow & \check{H}^p(A)\oplus\check{H}^p(B) & \longrightarrow & \check{H}^p(A\cap B) & \longrightarrow & \check{H}^{p+1}(A\cup B) & \longrightarrow \cdots \\
& \downarrow{\scriptstyle \cap x_{A\cup B}} & & \downarrow{\scriptstyle (\cap x_A)\oplus(\cap x_B)} & & \downarrow{\scriptstyle \cap x_{A\cap B}} & & \downarrow{\scriptstyle \cap x_{A\cup B}} & \\
\cdots \longrightarrow & H_q(X|A\cup B) & \longrightarrow & H_q(X|A)\oplus H_q(X|B) & \longrightarrow & H_q(X|A\cap B) & \longrightarrow & H_{q-1}(X|A\cup B) & \longrightarrow \cdots
\end{array}$$

in which the homology classes $x_A, x_B, x_{A\cap B}$ are restrictions of the class $x_{A\cup B}$ in the diagram

$$\begin{array}{ccccc}
 & & H_n(X, X-A) & & \\
 & \nearrow & & \searrow & \\
H_n(X, X - A\cup B) & & & & H_n(X, X - A\cap B) \\
 & \searrow & & \nearrow & \\
 & & H_n(X, X-B) & &
\end{array}$$