23   Hom and Lim

We will now develop more properties of the tensor product: its relationship to homomorphisms and to direct limits.

The tensor product arose in our study of bilinear maps. Even more natural are *linear maps*. Given a commutative ring R and two R-modules M and N, we can think about the collection of all R-linear maps from M to N. Not only does this set form an abelian group (under pointwise addition of homomorphisms); it forms an R-module, with

$$(rf)(y) = f(ry) = rf(y), \quad r \in R, y \in M.$$

The check that this is again an R-module homomorphism uses commutativity of R. We will write $\mathrm{Hom}_R(M, N)$, or just $\mathrm{Hom}(M, N)$, for this R-module.

Since $\mathrm{Hom}(M, N)$ is an R-module, we are entitled to think about what an R-module homomorphism into it is. Given

$$f : L \to \mathrm{Hom}(M, N)$$

we can define a new function

$$\hat{f} : L \times M \to N, \quad \hat{f}(x, y) = (f(x))(y) \in N.$$

You should check that this new function $\hat{f}$ is R-bilinear! So we get a natural map

$$\mathrm{Hom}(L, \mathrm{Hom}(M, N)) \to \mathrm{Hom}(L \otimes M, N).$$

Conversely, given a map $\hat{f} : L \otimes M \to N$ and $x \in L$, we can define $f(x) : M \to N$ by the same formula. These are inverse operations, so:

Lemma 23.1. *The natural map* $\mathrm{Hom}(L, \mathrm{Hom}(M, N)) \to \mathrm{Hom}(L \otimes M, N)$ *is an isomorphism.*

One says that $\otimes$ and Hom are *adjoint*, a word suggested by Sammy Eilenberg to Dan Kan, who first formulated this relationship between functors [7].

The second thing we will discuss is a generalization of one perspective on how the rational numbers are constructed from the integers – by a limit process: there are compatible maps in the diagram

and $\mathbf{Q}$ is the "universal," or "initial," abelian group you can map to.

We will formalize this process, using partially ordered sets as indexing sets. Recall from Lecture 3 that a *partially ordered set*, or *poset*, is a small category $\mathcal{I}$ such that $\#\mathcal{I}(i, j) \leq 1$ and the only isomorphisms are the identity maps. We will be interested in a particular class of posets.

Definition 23.2. A poset $(\mathcal{I}, \leq)$ is *directed* if for every $i, j \in \mathcal{I}$ there exists $k \in \mathcal{I}$ such that $i \leq k$ and $j \leq k$.

Example 23.3. This is a very common condition. A first example is the natural numbers $\mathbb{N}$ with $\leq$ as the order. Another example is the positive natural numbers, with $i \leq j$ if $i|j$. This is because $i, j|(ij)$. A topological example: if X is a space, A a subspace, and I is the set of open subsets of X containing A, directed by saying that $U \leq V$ if $U \supseteq V$. This is because an intersection of two opens is again open.

Definition 23.4. Let $\mathcal{I}$ be a directed set. An $\mathcal{I}$-*directed system* in a category $\mathcal{C}$ is a functor $\mathcal{I} \to \mathcal{C}$. This means that for every $i \in \mathcal{I}$ we are given an object $X_i \in \mathcal{C}$, and for every $i \leq j$ we are given a map $f_{i,j} : X_i \to X_j$, in such a way that $f_{i,i} = 1_{X_i}$ and if $i \leq j \leq k$ then $f_{i,k} = f_{j,k} \circ f_{i,j} : X_i \to X_k$.

Example 23.5. If $\mathcal{I} = (\mathbb{N}, \leq)$, then you get a "linear system" $X_0 \xrightarrow{f_{01}} X_1 \xrightarrow{f_{12}} X_2 \to \cdots$.

Example 23.6. Suppose $\mathcal{I} = (\mathbb{N}_{>0}, |)$, i.e., the second example above. You can consider $\mathcal{I} \to \mathbf{Ab}$, say assigning to each i the integers $\mathbf{Z}$, and $f_{ij} : \mathbf{Z} \xrightarrow{j/i} \mathbf{Z}$.

These directed systems can be a little complicated. But there's a simple one, namely the constant one.

Example 23.7. Let $\mathcal{I}$ be any directed system. Any object $A \in \mathcal{C}$ determines an $\mathcal{I}$-directed set, namely the constant functor $c_A : \mathcal{I} \to \mathcal{C}$.

Not every directed system is constant, but we can try to find a best approximating constant system. To compare systems, we need morphisms. $\mathcal{I}$-directed systems in $\mathcal{C}$ are functors $\mathcal{I} \to \mathcal{C}$. They are related by natural transformations, and those are the morphisms in the category of $\mathcal{I}$-directed systems. That is to say, a morphism is a choice of map $g_i : X_i \to Y_i$, for each $i \in \mathcal{I}$, such that

$$\begin{array}{ccc} X_i & \longrightarrow & X_j \\ \downarrow{\scriptstyle g_i} & & \downarrow{\scriptstyle g_j} \\ Y_i & \longrightarrow & Y_j \end{array}$$

commutes for all $i \leq j$.

Definition 23.8. Let $X : \mathcal{I} \to \mathcal{C}$ be a directed system. A *direct limit* is an object L and a map $X \to c_L$ that is initial among maps to constant systems. This means that given any other map to a constant system, say $X \to c_A$, there is a unique map $f : L \to A$ such that

$$\begin{array}{ccc} & & c_L \\ & \nearrow & \\ X & & \downarrow{\scriptstyle c_f} \\ & \searrow & \\ & & c_A \end{array}$$

commutes.

This is a "universal property." So two different direct limits are canonically isomorphic; but a directed system may fail to have a direct limit. For example, the linear directed systems we used to create the rational numbers exists in the category of finitely generated abelian groups; but $\mathbf{Q}$ is not finitely generated, and there's no finitely generated group that will serve as a direct limit of this system in the category of finitely generated abelian groups.

Example 23.9. Suppose we have an increasing sequence of subspaces, $X_0 \subseteq X_1 \subseteq \cdots \subseteq X$. This gives us a directed system of spaces, directed by the poset $(\mathbb{N}, \leq)$. It's pretty clear that as a *set* the direct limit of this system is the union of the subspaces. Saying that X is the direct limit of this directed system of spaces is saying first that X is the union of the X_i's, and second that the topology on X is determined by the topology on the subspaces; it's the "weak topology," characterized by the property that a map $f : X \to Y$ is continuous if and only if the restriction of f to each X_n is continuous. This is saying that a subset of X is open if and only if its intersection with each X_n is open in X. Our example is that a CW-complex is the direct limit of its skelata.

Direct limits may be constructed from the material of coproducts and quotients. So suppose $X : \mathcal{I} \to \mathcal{C}$ is a directed system. To construct the direct limit, begin by forming the coproduct over the elements of $\mathcal{I}$,

$$\coprod_{i \in \mathcal{I}} X_i \,.$$

There are maps $\mathrm{in}_i : X_i \to \coprod X_i$, but they are not yet compatible with the order relation in $\mathcal{I}$. Form a quotient of the coproduct to enforce that compatibility:

$$\varinjlim_{i \in \mathcal{I}} X_i = \left(\coprod_{i \in \mathcal{I}} X_i \right) / \sim$$

where $\sim$ is the equivalence relation generated by requiring that for any $i \in \mathcal{I}$ and any $x \in X_i$,

$$\mathrm{in}_i x \sim \mathrm{in}_j f_{ij}(x) \,.$$

The process of forming the coproduct and the quotient will depend upon the category you are working in, and may not be possible. In sets, coproduct is disjoint union and the quotient just forms equivalence classes. In abelian groups, the coproduct is the direct sum and to form the quotient you divide by the subgroup generated by differences.

Direct limits and the tensor product are nicely related, and the way to see that is to use the adjunction with Hom that we started with today.

Proposition 23.10. *Let $\mathcal{I}$ be a direct set, and let $M : \mathcal{I} \to \mathbf{Mod}_R$ be a $\mathcal{I}$-directed system of R-modules. There is a natural isomorphism*

$$(\varinjlim_{I} M_i) \otimes_R N \cong \varinjlim_{I} (M_i \otimes_R N) \,.$$

Proof. Let's verify that both sides satisfy the same universal property. A map from $(\varinjlim_{I} M_i) \otimes_R N$ to an R-module L is the same thing as a linear map $\varinjlim_{I} M_i \to \mathrm{Hom}_R(N, L)$. This is the same as a compatible family of maps $M_i \to \mathrm{Hom}_R(N, L)$, which in turn is the same as a compatible family of maps $M_i \otimes_R N \to L$, which is the same as a linear map $\varinjlim_{I} (M_i \otimes_R N) \to L$. $\qquad\square$

Here's a lemma that lets us identify when a map to a constant functor is a direct limit.

Lemma 23.11. *Let $X : \mathcal{I} \to \mathbf{Ab}$ (or $\mathbf{Mod}_R$). A map $f : X \to c_L$ (given by $f_i : X_i \to L$ for $i \in \mathcal{I}$) is the direct limit if and only if:*

1. *For every $x \in L$, there exists an i and an $x_i \in X_i$ such that $f_i(x_i) = x$.*

2. *Let $x_i \in X_i$ be such that $f_i(x_i) = 0$ in L. Then there exists some $j \geq i$ such that $f_{ij}(x_i) = 0$ in X_j.*

Proof. Straightforward. $\qquad\square$

Proposition 23.12. *The direct limit functor $\varinjlim_{I} : \mathrm{Fun}(\mathcal{I}, \mathbf{Ab}) \to \mathbf{Ab}$ is exact. In other words, if $X \xrightarrow{p} Y \xrightarrow{q} Z$ is an exact sequence of $\mathcal{I}$-directed systems (meaning that at every degree we get an exact sequence of abelian groups), then $\varinjlim_{I} X \to \varinjlim_{I} Y \to \varinjlim_{I} Z$ is exact.*

Proof. First of all, $qp : X \to Z$ is zero, which is to say that it factors through the constant zero object, so $\varinjlim_I X \to \varinjlim_I Z$ is certainly the zero map. Let $y \in \varinjlim_I Y$, and suppose y maps to 0 in $\varinjlim_I Z$. By condition (1) of Lemma 23.11, there exists i such that $y = f_i(y_i)$ for some $y_i \in Y_i$. Then $0 = q(y) = f_i q(y_i)$ because q is a map of direct systems. By condition (2), this means that there is $j \geq i$ such that $f_{ij} q(y_i) = 0$ in Z_j. So $q f_{ij} y_i = 0$, again because q is a map of direct systems. We have an element in Y_j that maps to zero under q, so there is some $x_j \in X_j$ such that $p(x_j) = y_j$. Then $f_j(x_j) \in \varinjlim_I X$ maps to y. $\qquad\square$

The exactness of the direct limit has many useful consequences. For example:

Corollary 23.13. *Let $i \mapsto C(i)$ be a directed system of chain complexes. Then there is a natural isomorphism*

$$\varinjlim_{i \in \mathcal{I}} H_*(C(i)) \to H_*(\varinjlim_{i \in \mathcal{I}} C(i)).$$

Putting together things we have just said:

Corollary 23.14. $H_*(X; \mathbf{Q}) = H_*(X) \otimes \mathbf{Q}$.

So we can redefine the Betti numbers of a space X as

$$\beta_n = \dim_{\mathbf{Q}} H_n(X; \mathbf{Q})$$

and discuss the Euler characteristic entirely in terms of the rational vector spaces making up the rational homology of X.

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