Compactly generated spaces

Definition. A space X is said to be compactly generated if it satisfies the following condition: A set A is open in X if $A \cap C$ is open in C for each compact subspace C of X.

Said differently, a space is compactly generated if its topology is coherent with the collection of compact subspaces of X. (See Notes G for a discussion of coherent topologies.) Many spaces are compactly generated; for instance, locally compact spaces are compactly generated, and so are first-countable spaces. (See Lemma 46.3.)

Compactly generated spaces are useful when studying various topologies on the space $\mathcal{C}(X,Y)$ of continuous functions $f : X \to Y$, but they occur in other contexts as well. Here we explore their relation to proper maps and to perfect maps.

Definition. A map $f : X \to Y$ is said to be proper if for every compact subspace C of Y, the subspace $f^{-1}(C)$ of X is compact.

Roughly speaking, f is proper if it does not collapse any subset of X that runs off to infinity onto a compact subspact of Y, which does not run off to infinity.

Theorem K.1. Let $f : X \to Y$ be a continuous map. If Y is a compactly generated Hausdorff space, and if f is proper, then f is a closed map.

Proof. Let A be a closed set in X. To show $f(A)$ is closed, we need only to show that $f(A) \cap C$ is closed in C for each compact subspace C of Y. Now

$$f(A) \cap C = f(f^{-1}(C) \cap A).$$

The space $f^{-1}(C)$ is compact because f is proper, so its closed subspace $f^{-1}(C) \cap A$ is also compact. The image of this set under f is compact, and is therefore closed in Y. $\square$

Corollary K.2. Let $f : X \to Y$ be continuous and injective. If f is proper, and Y is compactly generated Hausdorff, then f is an imbedding whose image is a closed subspace of Y.

Example 1. Let $f : [0,2\pi) \to \mathbb{R}^2$ be given by the equation $f(t) = (\cos t, \sin t)$. Then f is continuous and injective, and its image is the unit circle, which is closed in $\mathbb{R}^2$. However, f is not proper; the inverse image of the unit circle is not compact. And f is not an imbedding.

Definition. A map $f : X \to Y$ is said to be perfect if it is continuous, closed, and surjective, and if $f^{-1}(\{y\})$ is compact for each $y \in Y$.

Said differently, a perfect map is a closed quotient map such that the inverse image of each point is compact. Perfect maps have many special properties. For instance, if $f : X \to Y$ is perfect and Y is compact, then X is compact; the same result holds if "compact" is replaced by "paracompact." On the other hand, many "niceness" properties of X (such as the Hausdorff condition, regularity, local compactness, and second-countability, as well as the condition of being paracompact Hausdorff) are preserved by perfect maps. (See Exercise 12 of §26, Exercise 7 of §31, and Exercise 8 of §41.)

The relation between perfect maps and proper maps is given in the following theorem:

Theorem K.3. Every perfect map is proper. Conversely, let $f : X \to Y$ be continuous and surjective. If f is proper, and if Y is compactly generated Hausdorff, then f is perfect.

Proof. Suppose f is a perfect map. Let C be a compact subspace of Y; let $\mathcal{A}$ be an open cover of $f^{-1}(C)$. Given $y \in C$, the set $p^{-1}(\{y\})$ can be covered by finitely many elements of $\mathcal{A}$. Because f is a closed map, there is a neighborhood W of y such that $f^{-1}(W)$ is covered by these same elements of $\mathcal{A}$. (See Exercise 6 of §31.) We can cover C by finitely many such neighborhoods W; then their inverse images cover $f^{-1}(C)$.

Now we supose that f is continuous, surjective, and proper, and that Y is compactly generated Hausdorff. The fact that $f^{-1}(\{y\})$ is compact is immediate, since $\{y\}$ is compact. The fact that f is closed follows from Theorem K.1. □

It is an interesting fact that if X is not compactly generated, it may be given a (finer) topology that is compactly generated, and has exactly the same collection of compact subspaces:

Theorem K.4. Let X_T be a space with underlying set X and topology T. There is a unique topology C on X, finer than T, such that:

(i) If D is a subset of X, and if D is compact in the topology it inherits from X_T, or if D is compact in the topology it inherits from X_C, then these two topologies on D are the same.

(ii) X_C is compactly generated.

Proof. Let $\{C_\alpha\}$ be the family of compact subspaces of X_T. In view of Theorem G.4, there is a topology C on X, finer than T, such that each space C_α is a subspace of X_C and the topology of X_C is coherent with the subspaces C_α.

Suppose D is compact in the topology it inherits from X_T. Then in this topology, it is one of the spaces C_α; and as just noted, each space C_α is a subspace of X_C.

Now suppose D is compact in the topology it inherits from X_C. Because the identity map $i : X_C \rightarrow X_T$ is continuous, D is compact in the topology it inherits from X_T. Then the previous paragraph applies.

The space X_C is compactly generated. For by definition, U is open in X_C if and only if $U \cap D$ is open in D for each compact subspace D of X_T. But D is a compact subspace of X_T if and only if it is a compact subspace of X_C.

To prove uniqueness, let C' be any topology satisfying the conditions of the theorem. Because $X_{C'}$ is compactly generated, a set U is open in $X_{C'}$ if and only if $U \cap D$ is open in D for each compact subspace D of $X_{C'}$. By (i), D is a compact subspace of $X_{C'}$ if and only if it is a compact subspace of X_T. Therefore a set U is open in $X_{C'}$ if and only if it is open in X_C. □

The class of compactly generated spaces is an interesting one to explore. Like the class of normal spaces, it is not closed under the operations of taking subspaces or products. We shall show that if J is uncountable, then $\mathbb{R}^J$ is not compactly generated. It follows that the subspace $(0,1)^J$ of $[0,1]^J$ is not compactly generated, although $[0,1]^J$ is compact and thus compactly generated. It also follows that an arbitrary product of compactly generated spaces need not be compactly generated. (The same is true for finite products, but the required example is more complicated. See [D], p.249.)

Example 2. If J is uncountable, then $\mathbb{R}^J$ is not compactly generated. (This example is adapted from [Wd].)

Given $n \geq 1$, let A_n be the set of all points $\underline{x}$ of $\mathbb{R}^J$ such that $x_\alpha = 0$ for at most n values of α, and $x_\alpha = n$ for all other values of α. We show that each set A_n is closed in $\mathbb{R}^J$:

Let p be a point of $\mathbb{R}^J$ not in A_n. If $p_\beta \notin \{0,n\}$ for some β, let U be a neighborhood of p_β not containing 0 or n; then $\pi_\beta^{-1}(U)$ is a neighborhood of p disjoint from A_n. If $p_\alpha \in \{0,n\}$ for all α, then since $p \notin A_n$, there must be a finite set J_0 of indices containing more than n elements such that $p_\alpha = 0$ for $\alpha \in J_0$. Setting $U_\alpha = (-1,1)$ for $\alpha \in J_0$ and $U_\alpha = \mathbb{R}$ otherwise, we obtain a neighborhood $\prod U_\alpha$ of p disjoint from A_n.

We now show that if C is a compact subspace of $\mathbb{R}^J$, then C intersects only finitely many of the sets A_n. Since C is compact, so is $\pi_\alpha(C)$; therefore the latter is contained in some closed interval $[-n_\alpha, n_\alpha] = I_\alpha$ of $\mathbb{R}$. Then C lies in $\prod I_\alpha$; we show that $\prod I_\alpha$ intersects only finitely many sets A_n. Since J is uncountable, the map $\alpha \rightarrow n_\alpha$ of J into $\mathbb{Z}_+$ must map some infinite subset J_0 of J to a single integer N. It follows that A_n does not intersect $\prod I_\alpha$ if $n > N$. For if $\underline{x}$ belongs to A_n, then all but finitely many x_α are greater than N, while if $\underline{x}$ belongs to $\prod I_\alpha$, infinitely many $|x_\alpha|$ must be less than or equal to N.

Let T be the union of the sets A_n. We show that T is not closed in $\mathbb{R}^J$, but that $T \cap C$ is closed in C for every compact subspace C of $\mathbb{R}^J$.

It is easy to see that T is not closed in $\mathbb{R}^J$, for $\underline{0}$ is a limit point of T that is not in T. Given a basis element $\prod U_\alpha$ containing $\underline{0}$, let J_0 be the (finite) set of indices α for which $U_\alpha \neq \mathbb{R}$. If we set $x_\alpha = 0$ for $\alpha \in J_0$ and $x_\alpha = n$ otherwise (where n is the number of elements in J_0), we obtain a point $\underline{x}$ of T that lies in $\prod U_\alpha$.

It is also easy to see that $T \cap C$ is closed in C if C is a compact subspace of $\mathbb{R}^J$. For in this case $T \cap C$ is the union of finitely many sets of the form $A_n \cap C$; and since A_n is closed in $\mathbb{R}^J$, the set $A_n \cap C$ is closed in C.