Locally Euclidean Spaces

The basic objects of study in differential geometry are certain topological spaces called manifolds. One crucial property that manifolds possess is that they are locally just like euclidean space. Formally, this means that for each such space, there is an integer m such that each point of the space has a neighborhood that is homeomorphic to an open set of $\mathbb{R}^m$. We shall call such a space locally m-euclidean. If $m = 1$, we picture the space as a curve, and if $m = 2$, we picture it as a surface. Further conditions on the space are necessary, however, if these ways of picturing it are to be correct. But it is not immediately clear what these conditions should be.

We consider here several such conditions, before deciding which we shall impose in order that the space should be called a manifold. We consider them in order of increasing strength.

If X is locally m-euclidean, a few conditions hold immediately. The space X is, for instance, locally compact and locally metrizable, because $\mathbb{R}^m$ satisfies these conditions. Furthermore, X satisfies the T_1 axiom: If x is a point of X, we show $\{x\}$ is a closed set of X. Let y be another point of X. Choose a neighborhood U of y such that there is a homeomorphism h of U with an open set of $\mathbb{R}^m$. If U does not contain x, then U is a neighborhood of y disjoint from x. If U does contain x, then in the open set $h(U)$ of $\mathbb{R}^m$, we may choose a neighborhood W of $h(y)$ disjoint from $\{h(x)\}$; the set $h^{-1}(W)$ is then a neighborhood of y disjoint from $\{x\}$.

In the following discussion, X denotes a space that is locally m-euclidean.

Hausdorff. Although X satisfies the T_1 axiom, it need not be Hausdorff. Exercise 6 of §36 provides an example of a space that is locally 1-euclidean but not Hausdorff. It is commonly called the "line with two origins." Similar examples exist for every value of m.

If we impose the condition that X be Hausdorff, then X becomes a locally compact Hausdorff space, from which it follows that X is regular, indeed, completely regular. (See Exercise 7 of §33.)

Normal. However, the condition that X be Hausdorff does not imply that X is normal. There is a famous example, called the "Prüfer manifold," which is a locally 2-euclidean space that is Hausdorff but not normal. It is discussed in the J section of these notes.

Metrizable. In geometry, we should like our spaces to be reasonably nice. In fact, we should like them to be metrizable, or (even better) to be imbeddable as closed sets in some euclidean space $\mathbb{R}^N$. Normality is not sufficient, however, for either of these conditions to hold. There is an example called the "long line" that provides a counterexample. It is a normal, locally 1-euclidean space that is not metrizable. It is discussed in the C section of these notes.

Requiring X to be metrizable is in fact equivalent to requiring it to paracompact and Hausdorff. For every metrizable space is paracompact and Hausdorff (by Theorem 41.4), while a paracompact Hausdorff space that is locally metrizable is metrizable (by Theorem 42.1).

Hausdorff with a countable basis. Requiring X to be metrizable still does not ensure that it can be imbedded in euclidean space. For example, the product space $\mathbb{R} \times J$, where J is uncountable and has the discrete topology, is locally 1-euclidean and metrizable, but cannot be imbedded in euclidean space (since it has no countable basis).

Since X is locally compact, requiring X to be Hausdorff with a countable basis will ensure that it is metrizable, by the Urysohn metrization theorem. It will in fact ensure that X can be imbedded in $\mathbb{R}^N$ for some N. The compact case is easy and is dealt with in §36; the general case requires more work (See Exercise 7 of §50).

These facts lead to the following definition:

Definition. The locally m-euclidean space X is called an m-manifold if it is Hausdorff with a countable basis.

Now in fact the condition of being Hausdorff with a countable basis is not really much stronger than the condition of metrizability. For if the locally m-euclidean space X is metrizable, then each component of X is an m-manifold. This follows from Exercise 10 of §41, which is incorrect as it stands, but is made correct by replacing "paracompact" by "metrizable."

<u>Compact Hausdorff</u>. The strongest of the additional conditions one might impose on a locally m-euclidean space X is that it be compact and Hausdorff. These conditions imply that X is metrizable, by Exercise 7 of §34; this in turn implies (since X is compact) that X has a countable basis, by Exercise 3 of §34.

The condition of being compact Hausdorff is, however, stronger than we wish to impose in general, for it would exclude many useful spaces, such as $\mathbb{R}^m$ itself, from consideration. Compact manifolds are, however, particularly nice spaces to deal with; so we are delighted when compactness is satisfied. Proving theorems in the non-compact case often requires more effort. The imbedding theorems of §50 are a good example of this situation.