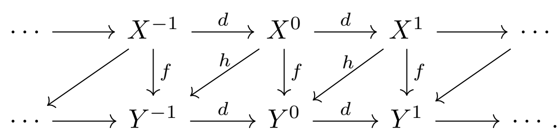
LECTURE 10

Homotopy, Quasi-Isomorphism, and Coinvariants

Please note that proofs of many of the claims in this lecture are left to Problem Set 5.

Recall that a sequence of abelian groups with differential d is a complex if $d^2 = 0$, $f\colon X \to Y$ is a morphism of chain complexes if $df = fd$, and h is a null-homotopy (of f) if $dh + hd = f$, which we illustrate in the following diagram:



The invariants of a chain complex are the homology groups

$$H^i(X) := \operatorname{Ker}(d\colon X^i \to X^{i+1}) / \operatorname{Im}(d\colon X^{i-1} \to X^i),$$

and for $f, g\colon X \rightrightarrows Y$, we say that $f \simeq g$, that is, f and g are homotopic, if and only if there exists a null-homotopy of $f - g$, which by Lemma 9.10, forces f and g to give the same map on cohomology.

For a finite group G and extension L/K of local fields with $G = \operatorname{Gal}(L/K)$, we have $\hat{H}^0(G, L^\times) = K^\times/\mathrm{N}L^\times$ by definition. Our goal is to show that $\hat{H}^0(G, L^\times) \simeq G^{\mathrm{ab}}$ canonically, i.e., the abelianization of G. Our plan for this lecture will be to define the Tate cohomology groups $\hat{H}^i$ for each $i \in \mathbb{Z}$ (which is more complicated for non-cyclic groups), and then use them to begin working towards a proof of this fact.

Recall that out basic principle was that, given a homotopy $h\colon f \simeq g$, f and g are now indistinguishable for all practical purposes (which we will take on faith). An application of this principle is the construction of cones or homotopy cokernels:

Claim 10.1. *If $f\colon X \to Y$ is a map of complexes, then* $\mathrm{hCoker}(f)$ *(a.k.a.* $\mathrm{Cone}(f)$*), characterized by the universal property that maps* $\mathrm{hCoker}(f) \to Z$ *of chain complexes are equivalent to maps $g\colon Y \to Z$ plus a null-homotopy h of $g \circ f\colon X \to Z$, exists.*

Proof. We claim that the following chain complex is $\mathrm{hCoker}(f)$:

$$\cdots \to X^0 \oplus Y^{-1} \to X^1 \oplus Y^0 \to X^2 \oplus Y^1 \to \cdots \tag{10.1}$$

with differential

$$X^{i+1} \oplus Y^i \ni \begin{pmatrix} x \\ y \end{pmatrix} \overset{d}{\longmapsto} \begin{pmatrix} -dx \\ f(x) + dy \end{pmatrix} \in X^{i+2} \oplus Y^{i+1},$$

which we note increases the degree appropriately. We may summarize this differential as a matrix $\left(\begin{smallmatrix} -d & 0 \\ f & d \end{smallmatrix}\right)$, and we note that it squares to zero as

$$\begin{pmatrix} -d & 0 \\ f & d \end{pmatrix} \begin{pmatrix} -d & 0 \\ f & d \end{pmatrix} = \begin{pmatrix} d^2 & 0 \\ -fd + df & d^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

by the definition of a morphism of chain complexes and because both X and Y are complexes.

We now check that this chain complex satisfies the universal property of hCoker(f). So suppose we have a map hCoker$(f) \to Z$, so that the diagram

$$\begin{array}{ccccccc} \cdots & \longrightarrow & X^{i+1} \oplus Y^i & \longrightarrow & X^{i+2} \oplus Y^{i+1} & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & Z^i & \longrightarrow & Z^{i+1} & \longrightarrow & \cdots \end{array}$$

commutes. If we write such a map as $(x, y) \mapsto h(x) + g(y)$, then this means

$$dh(x) + dg(y) = d(h(x) + g(y)) = h(-dx) + g(f(x) + dy) = -h(dx) + gf(x) + g(dy).$$

Taking $x = 0$ implies $dg = gd$, so we must have $dh + hd = g \circ f$, hence h is a null-homotopy of $g \circ f$, as desired. $\square$

Corollary 10.2. *The composition*

$$X \to Y \to \text{hCoker}(f)$$

is canonically null-homotopic (as an exercise, construct this null-homotopy explicitly!).

Example 10.3. Let

$$X := (\cdots \to 0 \to A \to 0 \to \cdots) \quad \text{and} \quad Y := (\cdots \to 0 \to B \to 0 \to \cdots)$$

for finite abelian groups A and B in degree 0, and let $f\colon A \to B$. Then

$$\text{hCoker}(f) = (\cdots \to 0 \to A \xrightarrow{f} B \to 0 \to \cdots),$$

with B in degree 0. Then we have

$$H^0\text{hCoker}(f) = \text{Coker}(f) \quad \text{and} \quad H^{-1}\text{hCoker}(f) = \text{Ker}(f),$$

so we see that the language of chain complexes generalizes prior concepts.

Notation 10.4. For a chain complex X, let $X[n]$ denote the *shift* of X by n places, that is, the chain complex with X^{i+n} in degree i, with the differential $(-1)^n d$ (where d denotes the differential for X). So for instance, $X[1] = \text{hCoker}(X \to 0)$. The content of this is that giving a null-homotopy of $0\colon X \to Y$ is equivalent to giving a map $X[1] \to Y$.

Lemma 10.5. *For all maps $f\colon X \to Y$, the sequence*

$$H^iX \to H^iY \to H^i\text{hCoker}(f)$$

is exact for all i.

Proof. The composition is zero by Lemma 9.10 because $X \to Y \to \text{hCoker}(f)$ is null-homotopic. To show exactness, let $y \in Y^i$ such that $dy = 0$, and suppose that its image in $H^i\text{hCoker}(f)$ is zero, so that

$$\begin{pmatrix} 0 \\ y \end{pmatrix} = \begin{pmatrix} -d & 0 \\ f & d \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} -d\alpha \\ f(\alpha) + d\beta \end{pmatrix}$$

for some $\alpha \in X^i$ with $d\alpha = 0$ and $\beta \in Y^{i-1}$. Then $f(\alpha) + d\beta = y$ implies $f(\alpha) = y$ in H^iY, as desired. □

Claim 10.6. *There is also a notion of the* homotopy kernel $\mathrm{hKer}(f)$*, defined by the universal property that maps* $Z \to \mathrm{hKer}(f)$ *are equivalent to maps* $Z \to X$ *plus the data of a null-homotopy of the composition* $Z \to X \to Y$*. In particular,* $\mathrm{hKer}(f) = \mathrm{hCoker}(f)[-1]$*.*

Example 10.7. Let $f\colon A \to B$ be a map of abelian groups (in degree 0 as before). Then

$$\mathrm{hCoker}(f) = (\cdots \to 0 \to A \xrightarrow{f} B \to 0 \to 0 \to \cdots)$$
$$\mathrm{hKer}(f) = (\cdots \to 0 \to 0 \to A \xrightarrow{f} B \to 0 \to \cdots),$$

where $\mathrm{hKer}(f)^0 = A$. The homotopy cokernel also recovers the kernel and cokernel in its cohomology.

Claim 10.8. *The composition*

$$X \xrightarrow{f} Y \to \mathrm{hCoker}(f)$$

is null-homotopic, so there exists a canonical map

$$X \to \mathrm{hKer}(Y \to \mathrm{hCoker}(f)),$$

where we refer to the latter term as "the mapping cylinder." This map is a homotopy equivalence.

Definition 10.9. A map $f\colon X \to Y$ is a *homotopy equivalence* if there exist a map $g\colon Y \to X$ and homotopies $gf \simeq \mathrm{id}_X$ and $fg \simeq \mathrm{id}_Y$, in which case we write $X \simeq Y$.

It is a *quasi-isomorphism* if $H^i(f)\colon H^i(X) \xrightarrow{\sim} H^i(Y)$ is an isomorphism for each i (i.e., X and Y are equal at the level of cohomology).

Claim 10.10. *If* $f\colon X \to Y$ *is a homotopy equivalence, then it is a quasi-isomorphism.*

Proof. This is an immediate consequence of Lemma 9.10, which ensures that f and g are inverses at the level of cohomology. □

Corollary 10.11. *Given* $f\colon X \to Y$*, there is a long exact sequence*

$$\cdots \to H^{i-1}\mathrm{hCoker}(f) \to H^iX \to H^iY \to H^i\mathrm{hCoker}(f) \to H^{i+1}X \to \cdots.$$

Proof. Letting g denote the map $Y \to \mathrm{hCoker}(f)$, the composition

$$Y \xrightarrow{g} \mathrm{hCoker}(f) \to \mathrm{hCoker}(g) = \mathrm{hKer}(g)[1] \simeq X[1]$$

is null-homotopic by Corollary 10.2, and the homotopy equivalence is by Claim 10.8. So by Lemma 10.5, the sequence

$$H^iY \to H^i\mathrm{hCoker}(f) \to H^iX[1] = H^{i+1}X$$

is exact; a further application of Lemma 10.5 shows the claim. □

Claim 10.12. *Suppose* $f^i\colon X^i \hookrightarrow Y^i$ *is injective for all* i*. Then* $\mathrm{hCoker}(f) \to Y/X$ *(i.e., the complex with* Y^i/X^i *in degree* i*) is a quasi-isomorphism.*

Example 10.13. If $f\colon A \hookrightarrow B$ is a map of abelian groups in degree 0, then the map $\mathrm{hCoker}(f) \to B/A$ looks like

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & A & \hookrightarrow & B & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & B/A & \longrightarrow & 0 & \longrightarrow & \cdots . \end{array}$$

It's easy to see that this is indeed a quasi-isomorphism. Note that there is a dual statement, that if f^i is surjective in each degree, then the homotopy kernel is quasi-isomorphic to the naive kernel.

Remark 10.14. If A is an associative algebra (e.g. $\mathbb{Z}$ or $\mathbb{Z}[G]$), then we can have chain complexes of A-modules

$$\cdots \to X^{-1} \xrightarrow{d} X^0 \xrightarrow{d} X^1 \to \cdots ,$$

where the X^i are A-modules and d is a map of A-modules. Here the cohomologies will also be A-modules.

Now, our original problem was to define Tate cohomology for a finite group G acting on some A. Note that

$$\hat{H}^0(G, A) = A^G/\mathrm{N}(A) = \mathrm{Coker}(\mathrm{N}\colon A \to A^G).$$

In fact, we can do better than $\mathrm{N}\colon A \to A^G$; the norm map factors through what we will call the coinvariants.

Definition 10.15. The *coinvariants* of A are $A_G := A/\sum_{g\in G}(g-1)A$, which satisfies the universal property that it is the maximal quotient of A with $gx = x$ holding for all $x \in A$ and $g \in G$.

Note that we can think of the invariants A^G as being the intersection of the kernels of each $(g-1)A$, so it is the maximal submodule of A for which $gx = x$ holds similarly. Then the norm map factors as

$$\begin{array}{ccc} A & \xrightarrow{\mathrm{N}} & A^G \\ {\scriptstyle \mathrm{N}}\downarrow & \nearrow & \\ A_G . & & \end{array}$$

Our plan is now to define derived (complex) versions of A_G and A^G called $A_{\mathrm{h}G} \xrightarrow{\mathrm{N}} A^{\mathrm{h}G}$, and Tate cohomology will be the homotopy cokernel of this map. The basic observation is that $\mathbb{Z}$ is a G-module (i.e. $\mathbb{Z}[G]$ acts on $\mathbb{Z}$) in a trivial way, with every $g \in G$ as the identity automorphism. If M is a G-module, then $M^G = \mathrm{Hom}_G(\mathbb{Z}, M)$ (because the image of 1 in M must be G-invariant and corresponds to the element of M^G) and $M_G = M \otimes_{\mathbb{Z}[G]} \mathbb{Z}$. Indeed, let $I \subseteq A$ be an ideal acting on M. Then $A/I \otimes_A M = M/IM$ by the right-exactness of tensor products. Here, $\mathbb{Z} = \mathbb{Z}[G]/I$, where I is the "augmentation ideal" generated by elements $g-1$ and therefore $M_G = M/I$ as desired.

Now we have the general problem where A is an associative algebra and M an associative A-module, and we would like the "derive" the functors $- \otimes_A M$ and $\mathrm{Hom}_A(M, -)$. These should take chain complexes of A-modules and produce complexes of abelian groups, preserving cones and quasi-isomorphisms. We'll begin working on this in the next lecture.