18.786 PROBLEM SET 6

(1) Let L/K be a finite Galois extension of nonarchimedean local fields, with Galois group G. We say this extension *admits an integral normal basis* if $\mathcal{O}_L \simeq \mathcal{O}_K[G]$ as an $\mathcal{O}_K[G]$-module.
  (a) Show that L/K admits an integral normal basis if and only if $\mathcal{O}_L/\mathcal{O}_L \cdot \mathfrak{p}_K$ is isomorphic to $k[G]$ as a $k[G]$-module, where $k := \mathcal{O}_K/\mathfrak{p}_K$.
  (b) Deduce that any tamely ramified extension admits an integral normal basis.
  (c) Let k_L denote the residue field of L, so $k_L = \mathcal{O}_L/\mathfrak{p}_L$ is a finite extension of k. Show that $\widehat{H}^0(G, k_L) = 0$ if and only if L/K is tamely ramified.
  (d) Deduce that there is an integral normal basis if and only if L/K is tamely ramified (this result is due to Noether).

(2) (a) For A an associative ring and M and N two A-modules, show that $\mathrm{Ext}^1_A(M, N)$ is canonically isomorphic to the set of isomorphism classes of extensions:

$$0 \to N \to E \to M \to 0.$$

Here an isomorphism of such extensions is a commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r] \arrow[d, equal] & E_1 \arrow[r] \arrow[d, "\simeq"] & M \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & N \arrow[r] & E_2 \arrow[r] & M \arrow[r] & 0. \end{tikzcd}$$

  (b) For $A = \mathbb{Z}[G]$, show that a group 1-cocycle $\varphi : G \to M$ is the same thing as an extension:

$$0 \to M \to E \to \mathbb{Z} \to 0$$

of G-modules (with G acting trivially on $\mathbb{Z}$) plus a lift of $1 \in \mathbb{Z}$ to an element of E. Show that different lifts define cocycles differing by coboundaries.
  (c) For A a ring, f a non-zero divisor and M an A-module, show that:[1]

$$\underline{\mathrm{Hom}}^{der}_A(A/f, M) \simeq \mathrm{hKer}(M \xrightarrow{f} M).$$

Deduce that $\mathrm{Ext}^i_A(A/f, M) = 0$ for $i > 1$, and $\mathrm{Ext}^1_A(A/f, M) = M/f$.
   Optional: use this to sketch a proof of the classification theorem for finitely generated modules over a PID.

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[1]Remember the definition of $\underline{\mathrm{Hom}}^{der}_A(N, M)$ (which is only well-defined up to quasi- isomorphism): it is $\underline{\mathrm{Hom}}_A(P, M)$ where $P \to N$ is some quasi-isomorphism with P projective.

(3) Suppose that H is a normal subgroup of G. For X a complex of G-modules, show that X^{hH} has a canonical[2] structure of complex of G/H-modules (as opposed to just being a complex of abelian groups), and show that $(X^{hH})^{h(G/H)} \simeq X^{hG}$.

(4) We will use the following definition.

Definition 1. A *(unital) DG*[3] *algebra* is a complex with a multiplication map:

$$m : A \otimes A \to A$$

(so this is a map of complexes) and an element $1 \in A^0$ such that $d(1) = 0$, and such that m satisfies associativity and unitality with respect to the element 1.

Below, we suppose that A is a DG algebra.

(a) For $a \in A^i$ and $b \in A^j$, let $ab = m(a \otimes b) \in A^{i+j}$. Show that:[4]

$$d(ab) = da \cdot b + (-1)^i a \cdot db.$$

(b) Show that $\oplus_{i \in \mathbb{Z}} H^i(A)$ inherits a canonical structure of associative algebra.

(c) Suppose that X is any complex. Show that $\underline{\mathrm{Hom}}(X, X)$ has a canonical DG algebra structure.

For $X = A$, show that the canonical map $A \to \underline{\mathrm{Hom}}(A, A)$ (corresponding to the multiplication $A \otimes A \to A$) a morphism of DG algebras.

(d) Suppose that there exists $\varepsilon \in A^{-1}$ such that $d(\varepsilon) = 1$. Show that A is homotopy equivalent to the zero complex.

(5) Suppose that B is an A-algebra, i.e., A is commutative and we have a map from A to the center of B.[5] Recall that the *bar complex* $\mathrm{Bar}_A(B)$ is the complex that in degree $-n$ ($n \geqslant 0$) is $B^{\otimes_A n+1}$, and with the differentials $d : B^{\otimes_A n+2} \to B^{\otimes_A n+1}$ given by:

$$b_0 \otimes \ldots \otimes b_{n+1} \mapsto$$
$$b_0 b_1 \otimes b_2 \otimes \ldots \otimes b_{n+1} - b_0 \otimes b_1 b_2 \otimes \ldots \otimes b_{n+1} + \ldots + (-1)^n b_0 \otimes b_1 \otimes \ldots \otimes b_n b_{n+1}.$$

(a) Show that the multiplication rule:

$$(b_0 \otimes \ldots \otimes b_{n+1}) \cdot (\beta_0 \otimes \ldots \otimes \beta_{m+1}) := b_0 \otimes \ldots \otimes b_n \otimes b_{n+1}\beta_0 \otimes \beta_1 \otimes \ldots \otimes \beta_{m+1}$$

makes $\mathrm{Bar}_A(B)$ into a DG algebra.

(b) Show that $1 = 0$ in $H^0(\mathrm{Bar}_A(B))$, and deduce that the bar complex is homotopy equivalent to zero.

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[2]This word is being used loosely here, since we're implicitly making choices that are only unique up to homotopy. Really, you should construct some complex of G/H-modules functorial with everything in sight and which computes X^{hH}, and which is not too stupid (say, in the sense that the second part of this problem is true).

[3]*DG* stands for *differential graded.* This phrase usually means "having something to do with chain complexes," since a chain complex is a $\mathbb{Z}$-graded abelian group equipped with a differential.

[4]Here concatenation and $\cdot$ indicate the same thing everywhere: multiplication.

[5]Here A and B are usual, i.e., non-DG algebras. (This is fundamentally inessential, but makes notation a little neater.)

(c) Show that the bar complex as a DG algebra satisfies a universal property: giving a map $\mathrm{Bar}_A(B) \to R$ (for R a DG algebra over[6] A) is equivalent to giving a map $B \to R$ of DG algebras over A (with B concentrated in degree 0) plus an element $\eta \in R^{-1}$ with $d\eta = 1$.

(6) (a) Suppose that we have a diagram:

$$\ldots \to X_2 \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0$$

of abelian groups.

Show that the inverse limit of this diagram is the kernel of the map:

$$\prod_{i=0}^{\infty} X_i \to \prod_{i=0}^{\infty} X_i$$
$$(x_0, x_1, x_2, \ldots) \mapsto (x_0 - f_1(x_1), x_1 - f_2(x_2), \ldots).$$

(b) Now suppose our diagram is of complexes $X_i = X_i^{\bullet}$. Define the *homotopy (inverse) limit* of this diagram as:

$$\operatorname*{holim}_i X_i := \mathrm{hKer}\Big(\prod_{i=0}^{\infty} X_i \to \prod_{i=0}^{\infty} X_i\Big)$$

where the map is as before.

Show that there is a canonical map:

$$H^n(\operatorname*{holim}_i X_i) \to \lim_i H^n(X_i)$$

for all $n \in \mathbb{Z}$. If all the maps $X_{i+1} \to X_i$ are termwise surjective (i.e., every map $X_{i+1}^j \to X_i^j$ is surjective) and all the maps $H^j(X_{i+1}) \to H^j(X_i)$ are surjective, show that the map you constructed is an isomorphism for every n.

(c) Show that this implies the statement from class a few weeks ago: given a diagram of short exact sequences of abelian groups:

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & M_1 \arrow[r] \arrow[d] & E_1 \arrow[r] \arrow[d] & N_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M_0 \arrow[r] & E_0 \arrow[r] & N_0 \arrow[r] & 0
\end{tikzcd}$$

with all maps $M_{i+1} \to M_i$ surjective, the induced map $\lim_i E_i \to \lim_i N_i$ is surjective as well.

[6]This means that R is a complex of A-modules, and its multiplication rule m descends to a map $m : R \otimes_A R \to R$. I.e., the multiplication is A-linear.

(d) For a complex Y, show that:

$$\underline{\mathrm{Hom}}(Y, \operatorname*{holim}_i X_i) = \operatorname*{holim}_i \underline{\mathrm{Hom}}(Y, X_i).$$

(e) Suppose now that the complexes and morphisms are of G-modules for G some group. Show that there is a canonical quasi-isomorphism:

$$\operatorname*{holim}_i (X_i^{hG}) \to (\operatorname*{holim}_i X_i)^{hG}.$$

(f) Suppose A is a ring, and P is a complex of A-modules.

For a diagram as above of complexes of A-modules, show that there is a canonical map:

$$P \bigotimes_A \operatorname*{holim}_i X_i \to \operatorname*{holim}_i \left(P \bigotimes_A X_i\right)$$

Show that this map is an isomorphism if P is a bounded above complex of finitely generated projective modules and if the complexes X_i are *uniformly bounded from above*, i.e., if there is an integer N such that $X_i^j = 0$ for all $j \geqslant N$ and all i. Carefully point out where you use the boundedness hypotheses.

(g) Deduce that homotopy coinvariants with respect to a finite group G commute with homotopy limits bounded uniformly from above. Deduce the same for Tate complexes. Reprove the claim from a few weeks ago: given a diagram $\ldots \to M_2 \to M_1 \to M_0$ of G-modules with all maps $M_{i+1} \to M_i$ surjective and $\widehat{H}^n(G, M_i) = 0$ for all n and i, then:

$$\widehat{H}^n(G, \lim_i M_i) = 0$$

for all n.

(h) Explain why we do not need to worry about the distinction between directed limits and homotopy directed limits (aka (directed) homotopy colimits). Indicate why group cohomology commutes with direct limits bounded uniformly from below for a finite group G, and similarly for Tate cohomology.

(7) For $G = \mathbb{Z}/n\mathbb{Z}$ and X a complex of G-modules, use our "nicer" resolutions to compute the Tate complex X^{tG}. More precisely, write down an explicit formula for a 2-periodic complex quasi-isomorphic to X^{tG}. What is the relationship between your formula and Problems (6g) and (6h)?

(8) Let $A = \mathbb{Z}/4\mathbb{Z}$ and consider the complex $P = P^\bullet$ with $P^i = A$ for all $i \in \mathbb{Z}$ and differential given by multiplication by 2. Show that P is not homotopy equivalent to the zero complex. Deduce that a complex of projective modules is not necessarily a projective complex.