15 Dirichlet's unit theorem

Let K be a number field. The two main theorems of classical algebraic number theory are:

- The class group $\operatorname{cl}\mathcal{O}_K$ is finite.
- The unit group $\mathcal{O}_K^\times$ is finitely generated.

We proved the first result in the previous lecture; in this lecture we will prove the second, due to Dirichlet. Dirichlet (1805–1859) died five years before Minkowski (1864–1909) was born, so he did not have Minkowski's lattice point theorem (Theorem 14.12) to work with. But we do, and this simplifies the proof considerably.

15.1 The group of Arakelov divisors of a global field

Let K be a global field. As in previous lectures, we use M_K to denote the set of places (equivalence classes of absolute values) of K. For each place $v \in M_K$ we use K_v to denote the completion of K with respect to v (a local field), and we have a normalized absolute value $\| \ \|_v \colon K_v \to \mathbb{R}_{\geq 0}$ defined by

$$\|x\|_v \coloneqq \frac{\mu(xS)}{\mu(S)},$$

where μ is a Haar measure on K_v and S is any measurable set of positive finite measure. This definition does not depend on the particular choice of μ or S; it is determined by the topology of K_v, which is an invariant of the place v (see Definition 13.17).

When K_v is nonarchimedean its topology is induced by a discrete valuation that we also denote v, and we use k_v to denote the residue field (the quotient of the valuation ring by its maximal ideal), which is a finite field (see Proposition 9.6). In Lecture 13 we showed that

$$\|x\|_v = \begin{cases} |x|_v = (\#k_v)^{-v(x)} & \text{if } v \text{ is nonarchimedean,}\\ |x|_{\mathbb{R}} & \text{if } K_v \simeq \mathbb{R},\\ |x|_{\mathbb{C}}^2 & \text{if } K_v \simeq \mathbb{C}.\end{cases}$$

While $\| \ \|_v$ is not always an absolute value (when $K_v \simeq \mathbb{C}$ it does not satisfy the triangle inequality), it is always multiplicative and defines a continuous homomorphism $K_v^\times \to \mathbb{R}_{>0}^\times$ of locally compact groups that is surjective precisely when v is archimedean.

Definition 15.1. Let K be a global field. A (multiplicative) *Arakelov divisor* (or *M_K-divisor*) is a sequence of positive real numbers $c = (c_v)$ indexed by $v \in M_K$ with all but finitely many $c_v = 1$ and $c_v \in \|K_v^\times\| \coloneqq \{\|x\|_v : x \in K_v^\times\}$.[1] The set of Arakelov divisors $\operatorname{Div} K$ forms an abelian group under pointwise multiplication $(c_v)(d_v) \coloneqq (c_v d_v)$. The multiplicative group $K^\times$ is canonically embedded in $\operatorname{Div} K$ via the map $x \mapsto (\|x\|_v)$, where it forms the subgroup of *principal Arakelov divisors*.

Remark 15.2. Many authors define $\operatorname{Div} K$ as an additive group by taking logarithms (for nonarchimedean places v, one replaces $c_v = (\#k_v)^{-v(c)}$ with the integer $v(c)$), as in [4] for example. The multiplicative convention we use here is due to Weil [5] and better suited to our application to the multiplicative group $\mathcal{O}_K^\times$.[2]

[1] When v is archimedean we have $\|K_v^\times\| = \mathbb{R}_{>0}$ and this constraint is automatically satisfied.
[2] Weil calls them *K-divisors* [5, p. 422], while Lang uses *M_K-divisors* [2].

Definition 15.3. Let K be a global field. The *size* of an Arakelov divisor c is the real number

$$\|c\| := \prod_{v\in M_K} c_v \in \mathbb{R}_{>0}.$$

The map $\operatorname{Div} K \to \mathbb{R}_{>0}^{\times}$ defined by $c \mapsto \|c\|$ is a group homomorphism that contains the subgroup of principal Arakelov divisors in its kernel (by the product formula, Theorem 13.21). Corresponding to each Arakelov divisor c is a subset $L(c)$ of K defined by

$$L(c) := \{x \in K : \|x\|_v \le c_v \text{ for all } v \in M_K\}.$$

and a nonzero fractional ideal of $\mathcal{O}_K$ defined by

$$I_c := \prod_{v\nmid\infty} \mathfrak{q}_v^{v(c)},$$

where $\mathfrak{q}_v := \{a \in \mathcal{O}_K : v(a) > 0\}$ is the prime ideal corresponding to the discrete valuation v that induces $\|\ \|_v$, and $v(c) := -\log_{\#k_v}(c_v) \in \mathbb{Z}$ (so $v(x) = v(c)$ if and only if $\|x\|_v = c_v$). We have $L(c) \subseteq I_c \subseteq K$, and the map $c \mapsto I_c$ defines a group homomorphism $\operatorname{Div} K \to \mathcal{I}_K$. Observe that to specify an Arakelov divisor c it suffices to specify the fractional ideal I_c and the real numbers $c_v > 0$ for $v|\infty$ (a finite set).

Remark 15.4. The quotient of $\operatorname{Div} K$ by its subgroup of principal divisors is denoted $\operatorname{Pic} K$. The homomorphism $\operatorname{Div} K \to \mathcal{I}_K$ sends principal Arakelov divisors to principal fractional ideals, and it follows that the ideal class group $\operatorname{cl}\mathcal{O}_K$ is a quotient of $\operatorname{Pic} K$. We have a commutative diagram

$$\begin{array}{ccc}
\operatorname{Div} K & \longrightarrow & \mathcal{I}_K \\
\downarrow & & \downarrow \\
\operatorname{Pic} K & \longrightarrow & \operatorname{cl}\mathcal{O}_K.
\end{array}$$

The Arakelov divisors of size 1 form a subgroup of $\operatorname{Div} K$ denoted $\operatorname{Div}^0 K$ that contains the subgroup of principal divisors and surjects onto $\mathcal{I}_K$ via the map $\operatorname{Div} K \to \mathcal{I}_K$ (we are free to choose any $I_c \in \mathcal{I}_K$ because we can always choose the c_v at infinite places to ensure $\|c\| = 1$). The quotient of $\operatorname{Div}^0 K$ by the subgroup of principal Arakelov divisors is the *Arakelov class group* $\operatorname{Pic}^0 K$, which admits the ideal class group $\operatorname{cl}\mathcal{O}_K$ as a finite quotient. See [4] for more background on Arakelov class groups and how to compute them.

Remark 15.5. The set $L(c)$ associated to an Arakelov divisor c is directly analogous to the *Riemann-Roch space*

$$L(D) := \{f \in k(X) : v_P(f) \ge -n_P \text{ for all closed points } P \in X\},$$

associated to a divisor $D \in \operatorname{Div} X$ of a smooth projective curve X/k, which is a k-vector space of finite dimension. Recall that a divisor is a formal sum $D = \sum n_P P$ over the closed points ($\operatorname{Gal}(\bar{k}/k)$-orbits) of the curve X with $n_P \in \mathbb{Z}$ and all but finitely many n_P zero.

If k is a finite field then $K = k(X)$ is a global field and there is a one-to-one correspondence between closed points of X and places of K, and a normalized absolute value $\|\ \|_P$ for each closed point P (indeed, one can take this as a definition). The constraint $v_P(f) \ge -n_P$ is equivalent to $\|f\|_P \le (\#k_P)^{n_P}$, where k_P is the residue field corresponding to P. If we put $c_P := (\#k_P)^{n_P}$ then $c = (c_P)$ is an Arakelov divisor with $L(c) = L(D)$. The Riemann-Roch space $L(D)$ is finite (since k is finite), and we will prove below that $L(c)$ is also finite (but note that when K is a number field the finite set $L(c)$ is not a vector space).

In §6.3 we described the divisor group $\operatorname{Div} X$ as the additive analog of the ideal group of the ring of integers $A = \mathcal{O}_K$, equivalently, the coordinate ring $A = k[X]$, of the global function field $K = k(X)$. When X is a smooth projective curve this is not a perfect analogy because divisors in $\operatorname{Div} X$ may include terms corresponding to "points at infinity" which do not correspond to fractional ideal of A. The group of Arakelov divisors $\operatorname{Div} K$ takes these infinite places into account and is a more exact analog of $\operatorname{Div} X$ when X is a smooth projective curve over a finite field.

We now specialize to the case where K is a number field. Recall that the absolute norm $\mathrm{N}(I)$ of a fractional ideal of $\mathcal{O}_K$ is the unique $t \in \mathbb{Q}_{>0}$ for which $N_{\mathcal{O}_K/\mathbb{Z}}(I) = (t)$. We have

$$\mathrm{N}(I_c) = \prod_{v\nmid\infty} \mathrm{N}(\mathfrak{q}_v)^{v(c)} = \prod_{v\nmid\infty} (\#k_v)^{v(c)} = \prod_{v\nmid\infty} c_v^{-1},$$

and therefore

$$\|c\| = \mathrm{N}(I_c)^{-1} \prod_{v|\infty} c_v, \tag{1}$$

We also define

$$R_c := \{x \in K_{\mathbb{R}} : |x|_v \le c_v \text{ for all } v|\infty\},$$

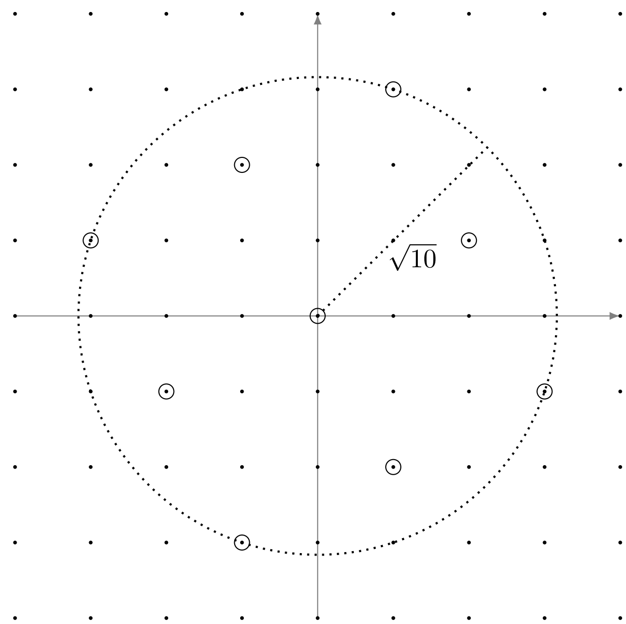
which we note is a compact, convex, symmetric subset of the real vector space

$$K_{\mathbb{R}} := K \otimes_{\mathbb{Q}} \mathbb{R} \simeq \mathbb{R}^r \times \mathbb{C}^s,$$

where r is the number of real places of K, and s is the number of complex places. If we view I_c and $L(c)$ as subsets of $K_{\mathbb{R}}$ via the canonical embedding $K \hookrightarrow K_{\mathbb{R}}$, then

$$L(c) = I_c \cap R_c.$$

Example 15.6. Let $K = \mathbb{Q}(i)$. The ideal $(2+i)$ lying above 5 is prime and corresponds to a finite place v_1, and there is a unique infinite place $v_2|\infty$ which is complex. Let $c_{v_1} = 1/5$, let $c_{v_2} = 10$, and set $c_v = 1$ for all other $v \in M_K$. We then have $I_c = (2+i)$ and the image of $L(c) = \{x \in (2+i) : |x|_\infty \le 10\}$ under the canonical embedding $K \hookrightarrow K_{\mathbb{R}} \simeq \mathbb{C}$ is the set of lattice points in the image of the ideal I_c that lie within the circle $R_c \subseteq K_{\mathbb{R}} \simeq \mathbb{C}$ of radius $\sqrt{10}$. Note that $\|\ \|_{v_2} = |\ |_{\mathbb{C}}^2$ is the square of the usual absolute value on $\mathbb{C}$, which is why the circle has radius $\sqrt{10}$ rather than 10.



The set $L(c)$ is clearly finite; it contains exactly 9 points.

Lemma 15.7. *Let c be an Arakelov divisor of a global field K. The set $L(c)$ is finite.*

Proof. We assume K is a number field; see Problem Set 7 for the function field case. The fractional ideal I_c is a lattice in $K_{\mathbb{R}}$ (under the canonical embedding $K \hookrightarrow K_{\mathbb{R}}$), and is thus a closed discrete subset of $K_{\mathbb{R}}$ (recall from Remark 14.4 that lattices are closed). In $K_{\mathbb{R}}$ we may view $L(c) = I_c \cap R_c$ as the intersection of a discrete closed set with a compact set, which is a compact discrete set and therefore finite. $\square$

Corollary 15.8. *Let K be a global field, and let μ_K denote the torsion subgroup of $K^\times$ (equivalently, the roots of unity in K). The group μ_K is finite and equal to the kernel of the map $K^\times \to \operatorname{Div} K$ defined by $x \mapsto (\|x\|_v)$; it is also the torsion subgroup of $\mathcal{O}_K^\times$.*

Proof. Each $\zeta \in \mu_K$ satisfies $\zeta^n = 1$ for some positive integer n. For every place $v \in M_K$ we have $\|\zeta^n\|_v = \|\zeta\|_v^n = 1$, and therefore $\|\zeta\|_v = 1$. It follows that $\mu_K \subseteq \ker(K^\times \to \operatorname{Div} K)$. Let c be the Arakelov divisor with $c_v = 1$ for all $v \in M_K$. Then $\ker(K^\times \to \operatorname{Div} K) \subseteq L(c)$ is a finite subgroup of $K^\times$ and is therefore contained in the torsion subgroup μ_K. Every element of μ_K is an algebraic integer (in fact a root of $x^n - 1$), so $\mu_K \subseteq \mathcal{O}_K^\times$. $\square$

It follows from Corollary 15.8 that for any global field K we have the following exact sequence of abelian groups

$$1 \longrightarrow \mu_K \longrightarrow K^\times \longrightarrow \operatorname{Div} K \longrightarrow \operatorname{Pic}_K \longrightarrow 1.$$

Proposition 15.9. *Let K be a number field with s complex places, define*

$$B_K := \left(\frac{2}{\pi}\right)^s \sqrt{|D_K|}.$$

If c is an Arakelov divisor of size $\|c\| > B_K$ then $L(c)$ contains an element of $K^\times$.

Proof. Our strategy is to apply Minkowski's lattice point theorem (see Theorem 14.12) to the convex symmetric set R_c and the lattice $I_c \subseteq K \subseteq K_{\mathbb{R}}$; we just need to show that if $\|c\| > B_K$ then the ratio of the Haar measure of R_c to the covolume of I_c exceeds 2^n, where $n = r + 2s$ is the degree of K (which is the real dimension of $K_{\mathbb{R}}$). As defined in §14.2, we normalize the Haar measure μ on the locally compact group $K_{\mathbb{R}} \simeq \mathbb{R}^r \times \mathbb{C}^s \simeq \mathbb{R}^n$ so that $\mu(S) = 2^s \mu_{\mathbb{R}^n}(S)$ for measurable $S \subseteq K_{\mathbb{R}}$. For each real place v, the constraint $\|x\|_v = |x|_{\mathbb{R}} \le c_v$ contributes a factor of $2c_v$ to $\mu(R_c)$, and for each complex place v the constraint $\|x\|_v = |x|_{\mathbb{C}}^2 \le c_v$ contributes a factor of πc_v (the area of a circle of radius $\sqrt{c_v}$). We may then compute

$$\begin{aligned}\frac{\mu(R_c)}{\operatorname{covol}(I_c)} &= \frac{2^s \mu_{\mathbb{R}^n}(R_c)}{\operatorname{covol}(I_c)} = \frac{2^s \left(\prod_{v \text{ real}} 2c_v\right)\left(\prod_{v \text{ complex}} \pi c_v\right)}{\operatorname{covol}(I_c)}\\ &= \frac{2^r (2\pi)^s \prod_{v|\infty} c_v}{\sqrt{|D_K|}\mathrm{N}(I_c)} = \frac{2^r(2\pi)^s}{\sqrt{|D_K|}}\|c\| = \frac{\|c\|}{B_K} 2^n > 2^n\end{aligned}$$

where we have used Corollary 14.16 and (1) in the second line. Theorem 14.12 implies that $L(c) = R_c \cap I_c$ contains a nonzero element (which lies in $K^\times \subseteq K_{\mathbb{R}}$, since $I_c \subseteq K \subseteq K_{\mathbb{R}}$). ☐

Remark 15.10. The bound in Proposition 15.9 can be turned into an asymptotic, that is, for $c \in \operatorname{Div} K$, as $\|c\| \to \infty$ we have

$$\#L(c) = \left(\frac{2^r(2\pi)^s}{\sqrt{|D_K|}} + o(1)\right)\|c\|. \tag{2}$$

This can be viewed as a multiplicative analog of the Riemann-Roch theorem for function fields, which states that for divisors $D = \sum n_P P$, as $\deg D := \sum n_P \to \infty$ we have

$$\dim L(D) = 1 - g + \deg D. \tag{3}$$

The nonnegative integer g is the *genus*, an important invariant of a function field that is often defined by (3); one could similarly use (2) to define the nonnegative integer $|D_K|$. For all sufficiently large $\|c\|$ the $o(1)$ error term will be small enough so that (2) uniquely determines $|D_K|$. Conversely, with a bit more work one can adapt the proofs of Lemma 15.7 and Proposition 15.9 to give a proof of the Riemann-Roch theorem for global function fields.

15.2 The unit group of a number field

Let K be a number field with ring of integers $\mathcal{O}_K$. The multiplicative group $\mathcal{O}_K^\times$ is the *unit group* of $\mathcal{O}_K$, and may also be called the unit group of K. Of course the unit group of the ring K is $K^\times$, but this is typically referred to as the multiplicative group of K.

As a ring, the finite étale $\mathbb{R}$-algebra $K_{\mathbb{R}} = K \otimes_{\mathbb{Q}} \mathbb{R}$ also has a unit group, and we have an isomorphism of topological groups[3]

$$K_{\mathbb{R}}^\times \simeq \prod_{v|\infty} K_v^\times \simeq \prod_{\text{real } v|\infty} \mathbb{R}^\times \prod_{\text{complex } v|\infty} \mathbb{C}^\times = (\mathbb{R}^\times)^r \times (\mathbb{C}^\times)^s.$$

[3]The additive group of $K_{\mathbb{R}}$ is isomorphic to $\mathbb{R}^n$ as a topological group (and $\mathbb{R}$-vector space), a fact we have used in our study of lattices in $K_{\mathbb{R}}$. But as topological rings $K_{\mathbb{R}} \simeq \mathbb{R}^r \times \mathbb{C}^s \not\simeq \mathbb{R}^n$ unless $s = 0$.

Writing elements of $K_{\mathbb{R}}^{\times}$ as vectors $x = (x_v)$ indexed by the infinite places v of K, we now define a surjective homomorphism of locally compact groups

$$\begin{aligned}\mathrm{Log}\colon K_{\mathbb{R}}^{\times} &\to \mathbb{R}^{r+s}\\ (x_v) &\mapsto (\log \|x_v\|_v).\end{aligned}$$

It is surjective and continuous because each of the maps $x_v \mapsto \log \|x_v\|_v$ is, and it is a group homomorphism because

$$\mathrm{Log}(xy) = (\log \|x_v y_v\|_v) = (\log \|x_v\|_v + \log \|y_v\|_v) = (\log \|x_v\|_v) + (\log \|y_v\|_v) = \mathrm{Log}\, x + \mathrm{Log}\, y;$$

here we have used the fact that the normalized absolute value $\| \ \|_v$ is multiplicative.

Recall from Corollary 13.7 that there is a one-to-one correspondence between the infinite places of K and the $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-orbits of $\mathrm{Hom}_{\mathbb{Q}}(K, \mathbb{C})$. For each $v|\infty$ let us now pick a representative σ_v of its corresponding $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-orbit in $\mathrm{Hom}_{\mathbb{Q}}(K, \mathbb{C})$; for real places v there is a unique choice for σ_v, while for complex places there are two choices, σ_v and its complex conjugate $\bar{\sigma}_v$. Regardless of our choices, we then have

$$\|x\|_v = \begin{cases} |\sigma_v(x)|_{\mathbb{R}} & \text{if } v|\infty \text{ is real} \\ |\sigma_v(x)\bar{\sigma}_v(x)|_{\mathbb{R}} & \text{if } v|\infty \text{ is complex.} \end{cases}$$

The absolute norm $\mathrm{N}\colon K^{\times} \to \mathbb{Q}_{>0}^{\times}$ extends naturally to a continuous homomorphism of locally compact groups

$$\begin{aligned}\mathrm{N}\colon K_{\mathbb{R}}^{\times} &\to \mathbb{R}_{>0}^{\times}\\ (x_v) &\mapsto \prod_{v|\infty} \|x_v\|_v\end{aligned}$$

which is compatible with the canonical embedding $K^{\times} \hookrightarrow K_{\mathbb{R}}^{\times}$. Indeed, we have

$$\mathrm{N}(x) = \left|\mathrm{N}_{K/\mathbb{Q}}(x)\right| = \left|\prod_{\sigma} \sigma(x)\right|_{\mathbb{R}} = \prod_{v|\infty} \|x\|_v.$$

We thus have a commutative diagram

$$\begin{tikzcd} K^{\times} \arrow[r, hook] \arrow[d, "\mathrm{N}"] & K_{\mathbb{R}}^{\times} \arrow[r, "\mathrm{Log}"] \arrow[d, "\mathrm{N}"] & \mathbb{R}^{r+s} \arrow[d, "\mathrm{T}"] \\ \mathbb{Q}_{>0}^{\times} \arrow[r, hook] & \mathbb{R}_{>0}^{\times} \arrow[r, "\log"] & \mathbb{R}, \end{tikzcd}$$

where $\mathrm{T}\colon \mathbb{R}^{r+s} \to \mathbb{R}$ is defined by $\mathrm{T}(x) = \sum_i x_i$. We may view Log as a map from $K^{\times}$ to $\mathbb{R}^{r+s}$ via the embedding $K^{\times} \hookrightarrow K_{\mathbb{R}}^{\times}$, and similarly view N as a map from $K^{\times}$ to $\mathbb{R}_{>0}^{\times}$.

We can succinctly summarize the commutativity of the above diagram by the identity

$$\mathrm{T}(\mathrm{Log}\, x) = \log \mathrm{N}(x),$$

which holds for all $x \in K^{\times}$, and all $x \in K_{\mathbb{R}}^{\times}$. The norm of a unit in $\mathcal{O}_K$ must be a unit in $\mathbb{Z}$, hence have absolute value 1. Thus $\mathcal{O}_K^{\times}$ lies in the kernel of the map $x \mapsto \log \mathrm{N}(x)$

and therefore also in the kernel of the map $x \mapsto \mathrm{T}(\operatorname{Log} x)$. It follows that $\operatorname{Log}(\mathcal{O}_K^\times)$ is a subgroup of the *trace zero hyperplane*

$$\mathbb{R}_0^{r+s} := \{x \in \mathbb{R}^{r+s} : \mathrm{T}(x) = 0\},$$

which we note is both a subgroup of $\mathbb{R}^{r+s}$, and an $\mathbb{R}$-vector subspace of dimension $r+s-1$. The proof of Dirichlet's unit theorem amounts to showing that $\operatorname{Log}(\mathcal{O}_K^\times)$ is a lattice in $\mathbb{R}_0^{r+s}$.

Proposition 15.11. *Let K be a number field with r real and s complex places, and let Λ_K be the image of the unit group $\mathcal{O}_K^\times$ in $\mathbb{R}_0^{r+s}$ under the* Log *map. The following hold:*

(1) *We have a split exact sequence of finitely generated abelian groups*

$$1 \to \mu_K \to \mathcal{O}_K^\times \xrightarrow{\operatorname{Log}} \Lambda_K \to 0;$$

(2) *Λ_K is a lattice in the trace zero hyperplane $\mathbb{R}_0^{r+s}$.*

Here μ_K is not a Haar measure, it denotes the group of roots of unity in K, all of which are clearly torsion elements of $\mathcal{O}_K^\times$, and any torsion element of $\mathcal{O}_K^\times$ is clearly a root of unity.

Proof. (1) We first show exactness. Let Z be the kernel of $\mathcal{O}_K^\times \xrightarrow{\operatorname{Log}} \Lambda_K$. Clearly $\mu_K \subseteq Z$, since $\Lambda_K \subseteq \mathbb{R}_0^{r+s}$ is torsion free. Let c be the Arakelov divisor with $I_c = \mathcal{O}_K$ and $c_v = 2$ for $v|\infty$, so that

$$L(c) = \{x \in \mathcal{O}_K : \|x\|_v \le 2 \text{ for all } v|\infty\}.$$

For $x \in \mathcal{O}_K^\times$ we have

$$x \in L(c) \Longleftrightarrow \operatorname{Log}(x) \in \operatorname{Log} R_c = \{z \in \mathbb{R}^{r+s} : z_i \le \log 2\}.$$

The set on the RHS includes the zero vector, thus $Z \subseteq L(c)$, which by Lemma 15.7 is a finite set. As a finite subgroup of $\mathcal{O}_K^\times$, we must have $Z \subseteq \mu_K$, so $Z = \mu_K$ and the sequence is exact (the map from $\mathcal{O}_K^\times$ to Λ_K is surjective by the definition of Λ_K).

We now show the sequence splits. Note that $\Lambda_K \cap \operatorname{Log}(R_c) = \operatorname{Log}\left(\mathcal{O}_K^\times \cap L(c)\right)$ is finite, since $L(c)$ is finite. It follows that 0 is an isolated point of Λ_K in $\mathbb{R}^{r+s}$, and in $\mathbb{R}_0^{r+s}$, so Λ_K is a discrete subgroup of the $\mathbb{R}$-vector space $\mathbb{R}_0^{r+s}$. It is therefore a free $\mathbb{Z}$-module of finite rank at most $r+s-1$, since it spans some subspace of $\mathbb{R}_0^{r+s}$ in which it is both discrete and cocompact, hence a lattice. It follows that $\mathcal{O}_K^\times$ is finitely generated, since it lies in a short exact sequence whose left and right terms are finitely generated (recall that μ_K is finite, by Corollary 15.8). By the structure theorem for finitely generated abelian groups, the sequence must split, since μ_K is the torsion subgroup of $\mathcal{O}_K^\times$.

(2) Having proved (1) it remains only to show that Λ_K spans $\mathbb{R}_0^{r+s}$. Let V be the subspace of $\mathbb{R}_0^{r+s}$ spanned by Λ_K and suppose for the sake of contradiction that $\dim V < \dim \mathbb{R}_0^{r+s}$. The orthogonal subspace $V^\perp$ then contains a unit vector u, and for every $\lambda \in \mathbb{R}_{>0}$ the open ball $B_{<\lambda}(\lambda u)$ does not intersect Λ_K. Thus $\mathbb{R}_0^{r+s}$ contains points arbitrarily far away from every point in Λ_K (with respect to any norm on $\mathbb{R}_0^{r+s} \subseteq \mathbb{R}^{r+s}$). To obtain a contradiction it is enough to show that there is a constant $M \in \mathbb{R}_{>0}$ such that for every $h \in \mathbb{R}_0^{r+s}$ there is an $\ell \in \Lambda_K$ for which $\|h - \ell\| := \max_i |h_i - \ell_i| < M$ (here we are using $\|\;\|$ to denote the sup norm on the $\mathbb{R}$-vector space $\mathbb{R}^{r+s}$).

Let us fix a real number $B > B_K$, where B_K is as in Proposition 15.9, so that for every $c \in \operatorname{Div} K$ with $\|c\| \ge B$ the set $L(c)$ contains a nonzero element, and fix a vector $b \in \mathbb{R}^{r+s}$

with nonnegative components b_i such that $\mathrm{T}(b) = \sum_i b_i = \log B$. Let $(\alpha_1), \ldots, (\alpha_m)$ be the list of all nonzero principal ideals with $\mathrm{N}(\alpha_j) \le B$ (by Lemma 14.20 this is a finite list). Let M be twice the maximum of $(r+s)B$ and $\max_j \|\operatorname{Log}(\alpha_j)\|$.

Now let $h \in \mathbb{R}_0^{r+s}$, and define $c \in \operatorname{Div} K$ by $I_c := \mathcal{O}_K$ and $c_v := \exp(h_i + b_i)$ for $v|\infty$, where i is the coordinate in $\mathbb{R}^{r+s}$ corresponding to v under the Log map. We have

$$\|c\| = \prod_v c_v = \exp\bigl(\sum_i (h_i + b_i)\bigr) = \exp \mathrm{T}(h+b) = \exp(\mathrm{T}(h) + \mathrm{T}(b)) = \exp \mathrm{T}(b) = B > B_K,$$

thus $L(c)$ contains a nonzero $\gamma \in I_c \cap K = \mathcal{O}_K$, and $g = \operatorname{Log}(\gamma)$ satisfies $g_i \le \log c_v = h_i + b_i$. We also have $\mathrm{T}(g) = \mathrm{T}(\operatorname{Log}\gamma) = \log \mathrm{N}(\gamma) \ge 0$, since $\mathrm{N}(\gamma) \ge 1$ for all nonzero $\gamma \in \mathcal{O}_K$. The vector $v := g - h \in \mathbb{R}^{r+s}$ satisfies $\sum_i v_i = \mathrm{T}(v) = \mathrm{T}(g) - \mathrm{T}(h) = \mathrm{T}(g) \ge 0$ and $v_i \le b_i \le B$ which together imply $|v_i| \le (r+s)B$, so $\|g - h\| = \|v\| \le M/2$. We also have

$$\log \mathrm{N}(\gamma) = \mathrm{T}(\operatorname{Log}(\gamma)) \le \mathrm{T}(h+b) = \mathrm{T}(b) = \log B,$$

so $\mathrm{N}(\gamma) \le B$ and $(\gamma) = (\alpha_j)$ for one of the α_j fixed above. Thus $\gamma/\alpha_j \in \mathcal{O}_K^\times$ is a unit, and

$$\ell := \operatorname{Log}(\gamma/\alpha_j) = \operatorname{Log}(\gamma) - \operatorname{Log}(\alpha_j) \in \Lambda_K$$

satisfies $\|g - \ell\| = \|\operatorname{Log}(\alpha_j)\| \le M/2$. We then have

$$\|h - \ell\| \le \|h - g\| + \|g - \ell\| \le M$$

as desired (by the triangle inequality for the sup-norm). □

Dirichlet's unit theorem follows immediately from Proposition 15.11.

Theorem 15.12 (Dirichlet's Unit Theorem). *Let K be a number field with r real and s complex places. Then $\mathcal{O}_K^\times \simeq \mu_K \times \mathbb{Z}^{r+s-1}$ is a finitely generated abelian group.*

Proof. The image of the torsion-free part of the unit group $\mathcal{O}_K^\times$ under the Log map is the lattice Λ_K in the trace-zero hyperplane $\mathbb{R}_0^{r+s}$, which has dimension $r+s-1$. □

We can restate this theorem in a more general form so that it applies to all global fields. As usual, when we consider global function fields we view them as extensions of $\mathbb{F}_q(t)$, with q chosen so that $K \cap \overline{\mathbb{F}}_q = \mathbb{F}_q$ and t chosen so that $K/\mathbb{F}_q(t)$ is separable.

Theorem 15.13 (Unit Theorem for Global Fields). *Let K/F be a finite separable extension, with $F = \mathbb{Q}$ or $F = \mathbb{F}_q(t)$, let $S \subseteq M_K$ be the set of places of K lying above the unique infinite place of F, and define $\mathcal{O}_K^\times := \{x \in K^\times : v(x) = 0$ for all $v \in M_K - S\}$. Then $\mathcal{O}_K^\times \simeq \mu_K \times \mathbb{Z}^{\#S-1}$ is a finitely generated abelian group.*

Proof. For $F = \mathbb{Q}$ we have $\#S = r + s$ and this is simply Dirichlet's unit theorem; for $F = \mathbb{F}_q(t)$, see [3, Prop. 14.1]. □

Remark 15.14. We should be careful in how we interpret 15.13 in the case $F = \mathbb{F}_q(t)$. By applying an automorphism of $\mathbb{F}_q(t)$ (replace t by $t - a$ for some $a \in \mathbb{F}_q$, say) we can move any degree-one place to infinity. This will change the group $\mathcal{O}_K^\times$ and may change the number of places of K above our new point at infinity. In contrast to the number field setting (where the place of $\mathbb{Q}$ at infinity is invariant because it is the only archimedean place) the ring $\mathcal{O}_K$ and the set S are not intrinsic to K in the function field setting; they depend on the choice of the separating element t used to construct the separable extension $K/\mathbb{F}_q(t)$.

Example 15.15. Let $K = \mathbb{Q}(\sqrt{d})$ be a quadratic field with $d \neq 1$ squarefree. If $d < 0$ then $r = 0$ and $s = 1$, in which case the unit group $\mathcal{O}_K^\times$ has rank 0 and $\mathcal{O}_K^\times = \mu_K$ is finite.

If $d > 0$ then $K = \mathbb{Q}(\sqrt{d}) \subseteq \mathbb{R}$ is a real quadratic field with $r = 2$ and $s = 0$, and the unit group $\mathcal{O}_K^\times$ has rank 1. The only torsion elements of $\mathcal{O}_K^\times \subseteq \mathbb{R}$ are ± 1, thus

$$\mathcal{O}_K^\times = \{\pm \epsilon^n : n \in \mathbb{Z}\},$$

for some $\epsilon \in \mathcal{O}_K^\times$ of infinite order. We may assume $\epsilon > 1$: if $\epsilon < 0$ then replace ϵ by $-\epsilon$, and if $\epsilon < 1$ then replace ϵ by ϵ^{-1} (we cannot have $\epsilon = 1 \in \mu_K$).

The assumption $\epsilon > 1$ uniquely determines ϵ. This follows from the fact that for $\epsilon > 1$ we have $|\epsilon^n| > |\epsilon|$ for all $n > 1$ and $|\epsilon^n| \leq 1$ for all $n \leq 0$.

This unique ϵ is the *fundamental unit* of $\mathcal{O}_K$ (and of K). To explicitly determine ϵ, let $D = \operatorname{disc} \mathcal{O}_K$ (so $D = d$ if $d \equiv 1 \bmod 4$ and $D = 4d$ otherwise). Every element of $\mathcal{O}_K$ can be uniquely written as

$$\frac{x + y\sqrt{D}}{2},$$

where x and Dy are integers of the same parity. In the case of a unit we must have $\mathrm{N}(\frac{x+y\sqrt{D}}{2}) = \pm 1$, equivalently,

$$x^2 - Dy^2 = \pm 4. \tag{4}$$

Conversely, any solution $(x, y) \in \mathbb{Z}^2$ to the above equation has x and Dy with the same parity and corresponds to an element of $\mathcal{O}_K^\times$. The constraint $\epsilon = \frac{x+y\sqrt{D}}{2} > 1$ forces $x, y > 0$. This follows from the fact that $\epsilon^{-1} = \frac{|x-y\sqrt{D}|}{2} < 1$, so $-2 < x - y\sqrt{D} < 2$, and adding and subtracting $x + y\sqrt{D} > 2$ shows $x > 0$ and $y > 0$ (respectively).

Thus we need only consider positive integer solutions (x, y) to (4). Among such solutions, $x_1 + y_1\sqrt{D} < x_2 + y_2\sqrt{D}$ implies $x_1 < x_2$, so the solution that minimizes x will give us the fundamental unit ϵ.

Equation (4) is a (generalized) *Pell equation*. Solving the Pell equation is a well-studied problem and there are a number of algorithms for doing so. The most well known uses continued fractions and is explored on Problem Set 7; this is not the most efficient method, but it is dramatically faster than an exhaustive search; see [1] for a comprehensive survey. A remarkable feature of this problem is that even when D is quite small, the smallest solution to (4) may be very large. For example, when $D = d = 889$ the fundamental unit is

$$\epsilon = \frac{26463949435607314430 + 887572376826907008\sqrt{889}}{2}.$$

15.3 The regulator of a number field

Let K be a number field with r real places and s complex places, and let $\mathbb{R}_0^{r+s}$ be the trace-zero hyperplane in $\mathbb{R}^{r+s}$. Choose any coordinate projection $\pi\colon \mathbb{R}^{r+s} \to \mathbb{R}^{r+s-1}$, and use the induced isomorphism $\mathbb{R}_0^{r+s} \xrightarrow{\sim} \mathbb{R}^{r+s-1}$ to endow $\mathbb{R}_0^{r+s}$ with a Euclidean measure. By Proposition 15.11, the image Λ_K of the unit group $\mathcal{O}_K^\times$ is a lattice in $\mathbb{R}_0^{r+s}$, and we can measure its covolume using the Euclidean measure on $\mathbb{R}_0^{r+s}$.

Definition 15.16. The *regulator* of a number field K is

$$R_K := \operatorname{covol}(\pi(\operatorname{Log}(\mathcal{O}_K^\times))) \in \mathbb{R}_{>0},$$

where $\pi\colon \mathbb{R}^{r+s} \to \mathbb{R}^{r+s-1}$ is any coordinate projection; the value of R_K does not depend on the choice of π, since we use π to normalize the Haar measure on $\mathbb{R}_0^{r+s} \simeq \mathbb{R}^{r+s-1}$. If $\epsilon_1, \ldots, \epsilon_{r+s-1}$ is a fundamental system of units (a $\mathbb{Z}$-basis for the free part of $\mathcal{O}_K^\times$), then R_K can be computed as the absolute value of the determinant of any $(r+s-1)\times(r+s-1)$ minor of the $(r+s)\times(r+s-1)$ matrix whose columns are the vectors $\operatorname{Log}(\epsilon_i) \in \mathbb{R}^{r+s}$.

Example 15.17. If K is a real quadratic field with absolute discriminant D and fundamental unit $\epsilon = \frac{x+y\sqrt{D}}{2}$, then $r+s=2$ and the product of the two real embeddings $\sigma_1(\epsilon), \sigma_2(\epsilon) \in \mathbb{R}$ is $\mathrm{N}(\epsilon) = \pm 1$. Thus $\log|\sigma_2(\epsilon)| = -\log|\sigma_1(\epsilon)|$ and

$$\operatorname{Log}(\epsilon) = (\log|\sigma_1(\epsilon)|, \log|\sigma_2(\epsilon)|) = (\log|\sigma_1(\epsilon)|, -\log|\sigma_1(\epsilon)|).$$

The 1×1 minors of the 2×1 transpose of $\operatorname{Log}(\epsilon)$ have determinant $\pm\log|\sigma_1(\epsilon)|$; the absolute value of the determinant is the same in both cases, and since we have require the fundamental unit to satisfy $\epsilon > 1$ (which forces a choice of embedding), the regulator of K is simply $R_K = \log\epsilon$.

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