Lecture 20
Periodic Continued Fractions, Quadratic Irrationalities

Review: $x = [a_0, a_1, \dots], x_0 = x, a_0 = x_0$, write $x = a_0 + \frac{1}{a_1 + \frac{1}{a_2 \cdots}} \Rightarrow a_1 = \frac{1}{x - a_0}$, at any point $x_n = [a_n, a_{n+1}, \dots]$, $x = x_0 = [a_0, a_1, \dots a_{n-1}, x_n]$, convergents $\frac{p_n}{q_n} = [a_0, a_1, \dots a_n]$. x is rational if and only if continued fraction is finite (terminates), and is quadratic irrational (ie., satisfies some quadratic equation) if and only if continued fraction is periodic.

Continuing proof (from last lecture). Proving that if x is a quadratic irrational, then continued fraction is periodic

Step 0:

$$x = \frac{a + \sqrt{b}}{c} \Rightarrow \frac{B_0 + \sqrt{d}}{C_0}$$

with B_0, C_0, d integers, $d > 0$, $C | d - B_0^2$

Step 1: Defined B_i, C_i by induction. $x_0 = x$, $a_i = \lfloor x_i \rfloor$, $x_i = \frac{B_i + \sqrt{d}}{C_i}$ defines B_i, C_i uniquely. $x_{i+1} = \frac{1}{x_i - a_i} \Rightarrow B_{i+1} = a_i C_i - B_i$, $C_{i+1} = \frac{d - B_{i+1}^2}{C_i}$, with $B_i, C_i \in \mathbb{Q}$.

Strategy: show all B_i, C_i are integers, then show are bounded, therefore repeat.

Step 2: By induction show B_i, C_i are integers and that $C_i | d - B_i^2$. For $i = 0$ it's obvious, as $B_0, C_0 \in \mathbb{Z}, C_0 | d - B_0^2$ by step 0. Easy to see that B_{i+1} is integer.

$$\begin{aligned} C_{i+1} &= \frac{d - B_{i+1}^2}{C_i} \\ &= \frac{d - (a_i C_i - B_i)^2}{C_i} \\ &= \frac{d - B_i^2 - a_i^2 C_i^2 + 2 a_i C_i B_i}{C_i} \\ &= \frac{d - B_i^2}{C_i} - a_i^2 C_i + 2 a_i B_i \in \mathbb{Z} \end{aligned}$$

Finally show that $C_{i+1} | d - B_{i+1}^2$ since $\frac{d - B_{i+1}^2}{C_{i+1}} = C_i$ is an integer.

Step 3: Check that $x_i = \frac{B_i + \sqrt{d}}{C_i}$ by induction. True for $i = 0$, $x_0 = \frac{B_0 + \sqrt{d}}{C_0}$. For

x_{i+1},

$$\begin{aligned}
x_{i+1} &= \frac{1}{x_i - a_i} \\
&= \frac{1}{\frac{B_i+\sqrt{d}}{C_i} - a_i} \\
&= \frac{C_i}{\sqrt{d} - (a_iC_i - B_i)} \\
&= \frac{C_i}{\sqrt{d} - B_{i+1}} \\
&= \frac{C_i(\sqrt{d} + B_{i+1})}{d - B_{i+1}^2} \\
&= \frac{\sqrt{d} + B_{i+1}}{C_{i+1}}
\end{aligned}$$

Step 4: Need to bound B_i, C_i. Let $y_i = \frac{B_i - \sqrt{d}}{C_i}$ be $\overline{x_i}$. We have $x = x_0 = \frac{x_n p_{n-1} + p_{n-2}}{x_n q_{n-1} + q_{n-2}}$ where $\{\frac{p_n}{q_n}\}$ are convergents to x. If we replace $\sqrt{d}$ by $-\sqrt{d}$ we get that $y_0 = \frac{y_n p_{n-1} + p_{n-2}}{y_n q_{n-1} + q_{n-2}}$. Solve for y_n

$$y_n = \frac{-(q_{n-2}y_0 - p_{n_2})}{q_{n-1}y_0 - p_{n-1}} = -\frac{q_{n-2}}{q_{n-1}} \left(\frac{y_0 - \frac{p_{n-2}}{q_{n-2}}}{y_0 - \frac{p_{n-1}}{q_{n-1}}} \right)$$

Now let $n \to \infty$, we get $-\frac{q_{n-2}}{q_{n-1}} \left(\frac{y_0 - x_0}{y_0 - x_0} \right)$, so for sufficiently large n the expression for y_n is negative.

Given $y_n = \frac{B_i - \sqrt{d}}{C_i}$, $x_n - y_n$ is positive, $x_n - y_n = \frac{2\sqrt{d}}{C_n} > 0$, then $C_n > 0$ for large enough n. Then $1 \leq C_n \leq C_n C_{n+1} = d - B_{n+1}^2 \leq d$, so C_n is bounded for large n (hence for all n). Also, $B_{n+1}^2 < B_{n+1}^2 + C_n C_{n+1} = d$, so $|B_{n+1}| < \sqrt{d}$ for large enough n, and so B_n is also bounded.

Step 5: There are only finitely many possibilities for (B_n, C_n), so there must be two natural numbers n and $n+k$ such that $(B_n, C_n) = (B_{n+k}, C_{n+k})$. Then

$$\begin{aligned}
x_n &= \frac{B_n + \sqrt{d}}{C_n} \\
&= \frac{B_{n+k} + \sqrt{d}}{C_{n+k}} \\
&= x_{n+k} \\
\Rightarrow a_n &= \lfloor x_n \rfloor \\
&= \lfloor x_{n+k} \rfloor \\
&= a_{n+k} \\
\Rightarrow B_{n+1} &= a_n C_n - B_n \\
&= a_{n+k} C_{n+k} - B_{n+k} \\
&= B_{n+k+1} \\
\Rightarrow C_{n+1} &= \frac{d - B_{n+1}^2}{C_n} \\
&= \frac{d - B_{n+k+1}^2}{C_{n+k}} \\
&= C_{n+k+1}
\end{aligned}$$

So $(B_{n+1}, C_{n+1}) = (B_{n+k+1}, C_{n+k+1})$, and so on, and so the representation $x_0 = x = [a_0, \dots a_{n-1}, \overline{a_n, a_{n+1}, \dots a_{n+k-1}}]$ is periodic.

Next, we want to understand what the continued fraction for $\sqrt{d}$ looks like for $d > 0$ not a square. One reason is to solve the Pell-Brahmagupta Equation, which is the diophantine equation $x^2 - dy^2 = 1$ for $x, y \in \mathbb{Z}$. If (x, y) is a positive solution to the P-B equation, then $(x + \sqrt{d}y)(x - \sqrt{d}y) = 1$, so since $x > \sqrt{d}y$,

$$\begin{aligned}
|x - \sqrt{d}y| &= \frac{1}{|x + \sqrt{d}y|} \\
\Rightarrow \left|\sqrt{d} - \frac{x}{y}\right| &= \frac{1}{y(x + \sqrt{d}y)} < \frac{1}{y(2\sqrt{d}y)}
\end{aligned}$$

$\Rightarrow \frac{x}{y}$ is an approximation to $\sqrt{d}$, which is at least as good as $\frac{1}{2\sqrt{d}y^2}$. If some $\frac{p}{q}$ approximates irrational α with error $\leq \frac{1}{2q^2}$ then it must be a convergent to α [proved in PSet 9], so all solutions to P-B equation must come from convergents $\frac{x}{y}$ of $\sqrt{d}$. ∎

Theorem 71. *Let x be a quadratic irrational, and $\overline{x}$ be its conjugate (ie., if $x = \frac{a+b\sqrt{d}}{c}$ with $a, b, c, d \in \mathbb{Z}$, then $\overline{x} = \frac{a-b\sqrt{d}}{c}$). The continued fraction of x is purely periodic (ie., $[\overline{a_0, a_1, \dots a_{n-1}}]$) if and only if $x > 1$ and $-1 < \overline{x} < 0$.*

Proof - Part 1. First suppose $x > 1$ and $-1 < \overline{x} < 0$. We know that continued fraction for x will repeat at some point, ie., there's an n-digit block that repeats and a "start point" m such that

$$x = [a_0, a_1, \dots a_{m-1}, \overline{a_m, a_{m+1}, \dots a_{m+n-1}}]$$

Want to show that we can take $m = 0$. We'll do this by downward induction - ie., by "advancing" m. We'll show that $a_{m-1} = a_{m-1+n}$.

We know that $a_i \geq 1$ for all i. So rewrite $x_{i+1} = \frac{1}{x_i - a_i}$ as $\frac{1}{x_{i+1}} = x_i - a_i$. Take conjugate

$$\frac{1}{\overline{x_{i+1}}} = \overline{x_i} - a_i$$

Now by induction, we'll show that $-1 < \overline{x_i} < 0$. For $i = 0$ this is by hypothesis. If we know for i then $\overline{x_i} - a_i < -1$, since $\overline{x_i} < 0$ and $a_i > 1$, and so $\frac{1}{\overline{x_{i+1}}} < -1$ which forces $-1 < \overline{x_{i+1}} < 0$, which completes the induction.

Then, since

$$-a_i - \frac{1}{\overline{x_{i+1}}} = -\overline{x_i} \in (0, 1)$$

we have $-\frac{1}{\overline{x_{i+1}}} \in (a_i, a_i + 1)$ and $\lfloor -\frac{1}{\overline{x_{i+1}}} \rfloor = a_i$. Now we know that $a_{m+k} = a_{m+k+n}$ for all $k \geq 0$.

$$x_m = [a_m, a_{m+1}, \dots] = [a_{m+n}, a_{m+n+1}, \dots] = x_{m+n}$$

so $\overline{x_m} = \overline{x_{m+n}}$.

$$a_{m-1} = \left\lfloor -\frac{1}{\overline{x_m}} \right\rfloor = \left\lfloor -\frac{1}{\overline{x_{m+n}}} \right\rfloor = a_{m+n-1}$$

therefore we can take $m = 0$, and so x is purely periodic. □

Proof. Suppose x is purely periodic, $x = [\overline{a_0, a_1, \dots a_{n-1}}]$. Want to show that $x > 1$ and $-1 < \overline{x} < 0$. For any x, $a_0 = a_n > 1 \Rightarrow x > 1$. So let's assume that $n \geq 4$ (can always take larger blocks if not). Now

$$x = [a_0, a_1, \dots a_{n-1}, x] = \frac{p_{n-1}x + p_{n-2}}{q_{n-1}x + q_{n-2}}$$

$$\Rightarrow q_{n-1}x^2 + (q_{n-2} - p_{n-1})x - p_{n-2} = 0 = f(x)$$

$\overline{x}$ is the other root. We know that $x > 1$, so it's enough to show that $f(x)$ has a root between -1 and 0. Do this by showing that $f(0)$ and $f(-1)$ have opposite signs.

$$\begin{aligned} f(0) &= -p_{n-2} < 0 \\ f(-1) &= q_{n-1} - q_{n-2} + p_{n-1} - p_{n-2} \\ &= (a_{n-1} - 1)q_{n-2} + q_{n-3} + (a_{n-1} - 1)p_{n-2} + p_{n-3} > 0 \end{aligned}$$

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