# 18.781 Problem Set 7

Thursday, April 26.
Collaboration is allowed and encouraged. However, your writeups should be your own, and you must note on the front the names of the students you worked with.
Extensions will only be given for extenuating circumstances.

In this problem set, $F_n$ denotes the $n$'th Fibonacci number.

1. Let $S$ be a set of $n+1$ integers selected from $1, 2, \ldots, 2n+1$. Prove that $S$ contains two relatively prime integers. Show that the result doesn't hold if only $n$ integers are chosen.

2. Prove that among any ten consecutive positive integers there is at least one which is coprime to the product of the others.

3. At a party, there are $n$ people, who each give their coat to a coat-check person. Calculate the number of ways in which the coats can be handed back, each person receiving one, so that *no one* receives their own coat.

4. Prove that $F_{m+n} = F_{m-1}F_n + F_mF_{n+1}$ for any positive integers $m$ and $n$. Then prove that $F_m | F_n$ if $m|n$.

5. Let $r(n)$ be the number of ways of writing a positive integer $n$ in the form
$$n = m_1 + m_2 \cdots + m_k$$
where $k$ and $m_1, \ldots, m_k$ are arbitrary positive integers.
   (a) Show that $r(n) = 1 + r(1) + r(2) + \cdots + r(n-1)$ for $n \geq 2$. Deduce that $r(n) = 2r(n-1)$ for $n \geq 2$ and therefore that $r(n) = 2^{n-1}$ for all positive integers $n$.
   (b) Establish this formula for $r(n)$ directly by a combinatorial argument.

6. Show that the number of ways of writing a positive integer $n$ in the form
$$n = m_1 + m_2 + \cdots + m_k$$
where $k$ is an arbitrary positive integer, and $m_1, \ldots, m_k$ are arbitrary *odd* positive integers, is $F_n$. (Hint: establish a recurrence and prove the result by induction).

7. Let $f(n)$ be the number of sequences $a_1, \ldots, a_n$ which can be constructed with each $a_i \in \{0, 1, 2\}$, and such that the sequence cannot contain two consecutive 0's or two consecutive 1's. Prove that $f(n)$ is the integer closest to $\frac{1}{2}(1+\sqrt{2})^{n+1}$.

8. Let $p > 5$ be a prime. Show that $F_p \equiv \left(\frac{p}{5}\right) \pmod{p}$. Show that $F_{p+1} \equiv 1 \pmod{p}$ if $p \equiv \pm 1 \pmod{5}$ and $F_{p+1} \equiv 0 \pmod{p}$ if $p \equiv \pm 2 \pmod{5}$. Conclude that if $p \equiv \pm 1 \pmod{5}$, then $p-1$ is a period of the sequence $F_n \pmod{p}$ (i.e. $F_n \equiv F_{n+p-1} \pmod{p}$ for all $n$).

9. (Bonus) Say that a set $S$ of positive integers has property $P$ if no element of $S$ is a multiple of another.

(a) Prove that there exists a subset $S$ of $\{1, \ldots, 2n\}$ of size $n$ with property $P$, but that no subset of size $n+1$ can have property $P$.

(b) Prove the same result for subsets of $\{1, \ldots, 2n-1\}$.

(c) How many elements are there in the largest subset $S$ of $\{1, \ldots, 2n-1\}$ having property $P$?

10. (Bonus) Let $C_n$ be the number of ways of fully parenthesizing a product of $n+1$ variables $x_0 \ldots x_n$, so that at each stage only two variables are multiplied. For example, for $n = 3$, we have the five ways

$$((xy)z)w, (x(yz))w, (xy)(zw), x(y(zw)), x((yz)w).$$

(a) Establish the recurrence

$$C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i} \text{ for } n \geq 0.$$

(b) Consider the generating function $C(z) = \sum_{n \geq 0} C_n z^n$. Show that

$$C(z) = 1 + zC(z)^2.$$

(c) Solve for $C(z)$, and calculate the coefficient of $z^n$, to show

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$