1.37. **Quantum traces.** Let $\mathcal{C}$ be a rigid monoidal category, V be an object in $\mathcal{C}$, and $a \in \mathsf{Hom}(V, V^{**})$. Define the left quantum trace

(1.37.1) $\qquad \mathrm{Tr}_V^L(a) := \mathrm{ev}_{V^*} \circ (a \otimes \mathrm{Id}_{V^*}) \circ \mathrm{coev}_V \in \mathsf{End}(\mathbf{1})$.

Similarly, if $a \in \mathsf{Hom}(V, {}^{**}V)$ then we can define the right quantum trace

(1.37.2) $\qquad \mathrm{Tr}_V^R(a) := \mathrm{ev}_{{}^{**}V} \circ (\mathrm{Id}_{{}^*V} \otimes a) \circ \mathrm{coev}_{{}^*V} \in \mathsf{End}(\mathbf{1})$.

In a tensor category over k, $\mathrm{Tr}^L(a)$ and $\mathrm{Tr}^R(a)$ can be regarded as elements of k.

When no confusion is possible, we will denote Tr_V^L by Tr_V.

The following proposition shows that usual linear algebra formulas hold for the quantum trace.

Proposition 1.37.1. *If $a \in \mathsf{Hom}(V, V^{**})$, $b \in \mathsf{Hom}(W, W^{**})$ then*

(1) $Tr_V^L(a) = Tr_{V^*}^R(a^*)$;
(2) $Tr_{V \oplus W}^L(a \oplus b) = Tr_V^L(a) + Tr_W^L(b)$ *(in additive categories)*;
(3) $Tr_{V \otimes W}^L(a \otimes b) = Tr_V^L(a)\, Tr_W^L(b)$;
(4) *If $c \in \mathsf{Hom}(V, V)$ then $Tr_V^L(ac) = Tr_V^L(c^{**}a)$, $Tr_V^R(ac) = Tr_V^R({}^{**}ca)$.*

Similar equalities to (2),(3) also hold for right quantum traces.

Exercise 1.37.2. Prove Proposition 1.37.1.

If $\mathcal{C}$ is a multitensor category, it is useful to generalize Proposition 1.37.1(2) as follows.

Proposition 1.37.3. *If $a \in \mathsf{Hom}(V, V^{**})$ and $W \subset V$ such that $a(W) \subset W^{**}$ then $Tr_V^L(a) = Tr_W^L(a) + Tr_{V/W}^L(a)$. That is, Tr is additive on exact sequences. The same statement holds for right quantum traces.*

Exercise 1.37.4. Prove Proposition 1.37.3.

1.38. **Pivotal categories and dimensions.**

Definition 1.38.1. Let $\mathcal{C}$ be a rigid monoidal category. A *pivotal structure* on $\mathcal{C}$ is an isomorphism of monoidal functors $a : \mathrm{Id} \xrightarrow{\sim} ?^{**}$.

That is, a pivotal structure is a collection of morphisms $a_X : X \xrightarrow{\sim} X^{**}$ natural in X and satisfying $a_{X \otimes Y} = a_X \otimes a_Y$ for all objects X, Y in $\mathcal{C}$.

Definition 1.38.2. A rigid monoidal category $\mathcal{C}$ equipped with a pivotal structure is said to be *pivotal*.

Exercise 1.38.3. (1) If a is a pivotal structure then $a_{V^*} = (a_V)^{*-1}$. Hence, $a_{V^{**}} = a_V^{**}$.

(2) Let $\mathcal{C} = \mathrm{Rep}(H)$, where H is a finite dimensional Hopf algebra. Show that pivotal structures on $\mathcal{C}$ bijectively correspond to group-like elements of H such that $gxg^{-1} = S^2(x)$ for all $x \in H$.

Let a be a pivotal structure on a rigid monoidal category $\mathcal{C}$.

Definition 1.38.4. The *dimension* of an object X with respect to a is $\dim_a(X) := \mathrm{Tr}(a_X) \in \mathsf{End}(\mathbf{1})$.

Thus, in a tensor category over k, dimensions are elements of k. Also, it follows from Exercise 1.38.3 that $\dim_a(V) = \dim_a(V^{**})$.

Proposition 1.38.5. *If $\mathcal{C}$ is a tensor category, then the function $X \mapsto \dim_a(X)$ is a character of the Grothendieck ring $\mathsf{Gr}(\mathcal{C})$.*

Proof. Proposition 1.37.3 implies that $\dim_a$ is additive on exact sequences, which means that it gives rise to a well-defined linear map from $\mathsf{Gr}(\mathcal{C})$ to k. The fact that this map is a character follows from the obvious fact that $\dim_a(\mathbf{1}) = 1$ and Proposition 1.37.1(3). $\square$

Corollary 1.38.6. *Dimensions of objects in a pivotal finite tensor category are algebraic integers in k.* [12]

Proof. This follows from the fact that a character of any ring that is finitely generated as a $\mathbb{Z}$-module takes values in algebraic integers. $\square$

1.39. **Spherical categories.**

Definition 1.39.1. A pivotal structure a on a tensor category $\mathcal{C}$ is *spherical* if $\dim_a(V) = \dim_a(V^*)$ for any object V in $\mathcal{C}$. A tensor category is *spherical* if it is equipped with a spherical structure.

Since $\dim_a$ is additive on exact sequences, it suffices to require the property $\dim_a(V) = \dim_a(V^*)$ only for simple objects V.

Theorem 1.39.2. *Let $\mathcal{C}$ be a spherical category and V be an object of $\mathcal{C}$. Then for any $x \in \mathsf{Hom}(V, V)$ one has $Tr_V^L(a_V x) = Tr_V^R(x a_V^{-1})$.*

Proof. We first note that $\mathrm{Tr}_X^R(a_X^{-1}) = \dim_a(X^*)$ for any object X by Proposition 1.37.1(1) and Exercise 1.38.3(1). Now let us prove the proposition in the special case when V is semisimple. Thus $V = \oplus_i Y_i \otimes V_i$, where V_i are vector spaces and Y_i are simple objects. Then $x =$

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[12]If k has positive characteristic, by an algebraic integer in k we mean an element of a finite subfield of k.

$\oplus_i x_i \otimes \mathrm{Id}_{V_i}$ with $x_i \in \mathsf{End}_k(Y_i)$ and $a = \oplus \mathrm{Id}_{Y_i} \otimes a_{V_i}$ (by the functoriality of a). Hence

$$\begin{aligned}
\mathrm{Tr}_V^L(ax) &= \sum \mathrm{Tr}(x_i) \dim(V_i), \\
\mathrm{Tr}_V^R(xa^{-1}) &= \sum \mathrm{Tr}(x_i) \dim(V_i^*).
\end{aligned}$$

This implies the result for a semisimple V.

Consider now the general case. Then V has the coradical filtration

(1.39.1) $$0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_n = V$$

(such that V_{i+1}/V_i is a maximal semisimple subobject in V/V_i). This filtration is preserved by x and by a (i.e., $a : V_i \to V_i^{**}$). Since traces are additive on exact sequences by Proposition 1.37.3, this implies that the general case of the required statement follows from the semisimple case. $\qquad\square$

Exercise 1.39.3. (i) Let $\mathrm{Aut}_\otimes(\mathrm{Id}_\mathcal{C})$ be the group of isomorphism classes of monoidal automorphisms of a monoidal category $\mathcal{C}$. Show that the set of isomorphism classes of pivotal structures on $\mathcal{C}$ is a torsor over $\mathrm{Aut}_\otimes(\mathrm{Id}_\mathcal{C})$, and the set of isomorphism classes of spherical structures on $\mathcal{C}$ is a torsor over the subgroup $\mathrm{Aut}_\otimes(\mathrm{Id}_\mathcal{C})_2$ in $\mathrm{Aut}_\otimes(\mathrm{Id}_\mathcal{C})$ of elements which act by ± 1 on simple objects.

1.40. **Semisimple multitensor categories.** In this section we will more closely consider semisimple multitensor categories which have some important additional properties compared to the general case.

1.41. **Isomorphism between V^{**} and V.**

Proposition 1.41.1. *Let $\mathcal{C}$ be a semisimple multitensor category and let V be an object in $\mathcal{C}$. Then $^*V \cong V^*$. Hence, $V \cong V^{**}$.*

Proof. We may assume that V is simple.

We claim that the unique simple object X such that $\mathsf{Hom}(\mathbf{1}, V \otimes X) \neq 0$ is V^*. Indeed, $\mathsf{Hom}(\mathbf{1}, V \otimes X) \cong \mathsf{Hom}(^*X, V)$ which is non-zero if and only if $^*X \cong V$, i.e., $X \cong V^*$. Similarly, the unique simple object X such that $\mathsf{Hom}(V \otimes X, \mathbf{1}) \neq 0$ is $^*V$. But since $\mathcal{C}$ is semisimple, $\dim_k \mathsf{Hom}(\mathbf{1}, V \otimes X) = \dim_k \mathsf{Hom}(V \otimes X, \mathbf{1})$, which implies the result. $\qquad\square$

Remark 1.41.2. As noted in Remark 1.27.2, the result of Proposition 1.41.1 is false for non-semisimple categories.

Remark 1.41.3. Proposition 1.41.1 gives rise to the following question.

Question 1.41.4. Does any semisimple tensor category admit a pivotal structure? A spherical structure?

This is the case for all known examples. The general answer is unknown to us at the moment of writing (even for ground fields of characteristic zero).

Proposition 1.41.5. *If $\mathcal{C}$ is a semisimple tensor category and $a : V \xrightarrow{\sim} V^{**}$ for a simple object V then $Tr(a) \neq 0$.*

Proof. $Tr(a)$ is the composition morphism of the diagram $\mathbf{1} \to V \otimes V^* \to \mathbf{1}$ where both morphisms are non-zero. If the composition morphism is zero then there is a non-zero morphism $(V \otimes V^*)/\mathbf{1} \to \mathbf{1}$ which means that the $[V \otimes V^* : 1] \geq 2$. Since $\mathcal{C}$ is semisimple, this implies that $\dim_k \mathsf{Hom}(1, V \otimes V^*)$ is at least 2. Hence, $\dim_k \mathsf{Hom}(V, V) \geq 2$ which contradicts the simplicity of V. $\square$

Remark 1.41.6. The above result is false for non-semisimple categories. For example, let $\mathcal{C} = \mathrm{Rep}_k(GL_p(\mathbb{F}_p))$, the representation category of the group $GL_p(\mathbb{F}_p)$ over a field k of characteristic p. Let V be the p dimensional vector representation of $GL_p(\mathbb{F}_p)$ (which is clearly irreducible). Let $a : V \to V^{**}$ be the identity map. Then $Tr(a) = \dim_k(V) = p = 0$ in k.

1.42. **Grothendieck rings of semisimple tensor categories.**

Definition 1.42.1. (i) A $\mathbb{Z}_+-basis$ of an algebra free as a module over $\mathbb{Z}$ is a basis $B = \{b_i\}$ such that $b_i b_j = \sum_k c_{ij}^k b_k$, $c_{ij}^k \in \mathbb{Z}_+$.
(ii) A $\mathbb{Z}_+-ring$ is an algebra over $\mathbb{Z}$ with unit equipped with a fixed $\mathbb{Z}_+-$basis.

Definition 1.42.2. (1) A $\mathbb{Z}_+-$ring A with basis $\{b_i\}_{i \in I}$ is called a *based ring* if the following conditions hold
[a] There exists a subset $I_0 \subset I$ such that $1 = \sum_{i \in I_0} b_i$.
[b] Let $\tau : A \to \mathbb{Z}$ be the group homomorphism defined by

$$(1.42.1) \qquad \tau(b_i) = \begin{cases} 1 & \text{if} \quad i \in I_0 \\ 0 & \text{if} \quad i \notin I_0 \end{cases}$$

There exists an involution $i \mapsto i^*$ of I such that induced map $a = \sum_{i \in I} a_i b_i \mapsto a^* = \sum_{i \in I} a_i b_{i^*}$, $a_i \in \mathbb{Z}$ is an anti-involution of ring A and such that

$$(1.42.2) \qquad \tau(b_i b_j) = \begin{cases} 1 & \text{if} \quad i = j^* \\ 0 & \text{if} \quad i \neq j^*. \end{cases}$$

(2) A *unital $\mathbb{Z}_+$-ring* is a $\mathbb{Z}_+$-ring A such that 1 belongs to the basis.

(3) A *multifusion ring* is a based ring of finite rank. A *fusion ring* is a unital based ring of finite rank.

Remark 1.42.3. (1) It follows easily from definition that $i, j \in I_0$, $i \neq j$ implies that $b_i^2 = b_i$, $b_i b_j = 0$, $i^* = i$.

(2) It is easy to see that for a given $\mathbb{Z}_+-$ring A, being a (unital) based ring is a *property*, not an additional structure.

(3) Note that any $\mathbb{Z}_+$-ring is assumed to have a unit, and is not necessarily a unital $\mathbb{Z}_+$-ring.

Proposition 1.42.4. *If $\mathcal{C}$ is a semisimple multitensor category then $\mathsf{Gr}(\mathcal{C})$ is a based ring. If $\mathcal{C}$ is semisimple tensor category then $\mathsf{Gr}(\mathcal{C})$ is a unital based ring. If $\mathcal{C}$ is a (multi)fusion category, then $\mathrm{Gr}(\mathcal{C})$ is a (multi)fusion ring.*

Proof. The $\mathbb{Z}_+$-basis in $\mathsf{Gr}(\mathcal{C})$ consists of isomorphism classes of simple objects of $\mathcal{C}$. The set I_0 consists of the classes of simple subobjects of $\mathbf{1}$. The involution $*$ is the duality map (by Proposition 1.41.1 it does not matter whether to use left or right duality). This implies the first two statements. The last statement is clear. $\qquad\square$

Example 1.42.5. Let $\mathcal{C}$ be the category of finite dimensional representations of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$. Then the simple objects of this category are irreducible representations V_m of dimension $m + 1$ for $m = 0, 1, 2, \ldots$; $V_0 = \mathbf{1}$. The Grothendieck ring of $\mathcal{C}$ is determined by the well-known Clebsch-Gordon rule

$$(1.42.3) \qquad V_i \otimes V_j = \bigoplus_{l=|i-j|, i+j-l\in 2\mathbb{Z}}^{i+j} V_l.$$

The duality map on this ring is the identity. The same is true if $\mathcal{C} = \mathrm{Rep}(U_{\mathsf{q}}(\mathfrak{sl}_2))$ when q is not a root of unity, see [K].

Let $\mathcal{C}$ be a semisimple multitensor category with simple objects $\{X_i\}_{i\in I}$. Let I_0 be the subset of I such that $\mathbf{1} = \oplus_{i\in I_0} X_i$. Let $H_{ij}^l := \mathsf{Hom}(X_l, X_i \otimes X_j)$ (if $X_p \in \mathcal{C}_{ij}$ with $p \in I$ and $i, j \in I_0$, we will identify spaces H_{pi}^p and H_{ip}^p with k using the left and right unit morphisms).

We have $X_i \otimes X_j = \bigoplus_l H_{ij}^l \otimes X_l$. Hence,

$$\begin{aligned}
(X_{i_1} \otimes X_{i_2}) \otimes X_{i_3} &\cong \bigoplus_{i_4} \bigoplus_j H_{i_1 i_2}^j \otimes H_{j i_3}^{i_4} \otimes X_{i_4} \\
X_{i_1} \otimes (X_{i_2} \otimes X_{i_3}) &\cong \bigoplus_{i_4} \bigoplus_l H_{i_1 l}^{i_4} \otimes H_{i_2 i_3}^l \otimes X_{i_4}.
\end{aligned}$$

Thus the associativity constraint reduces to a collection of linear isomorphisms

$$(1.42.4) \qquad \Phi^{i_4}_{i_1 i_2 i_3} : \bigoplus_j H^j_{i_1 i_2} \otimes H^{i_4}_{j i_3} \cong \bigoplus_l H^{i_4}_{i_1 l} \otimes H^l_{i_2 i_3}.$$

The matrix blocks of these isomorphisms,

$$(1.42.5) \qquad (\Phi^{i_4}_{i_1 i_2 i_3})_{jl} : H^j_{i_1 i_2} \otimes H^{i_4}_{j i_3} \to H^{i_4}_{i_1 l} \otimes H^l_{i_2 i_3}$$

are called $6j$-*symbols* because they depend on six indices.

Example 1.42.6. Let $\mathcal{C}$ be the category of finite dimensional representations of the Lie algebra $sl_2(\mathbb{C})$. Then the spaces H^l_{ij} are 0- or 1-dimensional. In fact, it is obvious from the Clebsch-Gordan rule that the map $(\Phi^{i_4}_{i_1 i_2 i_3})_{jl}$ is a map between nonzero (i.e., 1-dimensional) spaces if and only if the numbers i_1, i_2, i_3, i_4, j, l are edge lengths of a tetrahedron with faces corresponding to the four H-spaces ($i_1 i_2 j$, $j i_3 i_4, i_1 l i_4$, $i_2 i_3 l$, such that the perimeter of every face is even (this tetrahedron is allowed to be in the Euclidean 3-space, Euclidean plane, or hyperbolic 3-space, so the only conditions are the triangle inequalities on the faces). In this case, the $6j$-symbol can be regarded as a number, provided we choose a basis vector in every non-zero H^l_{ij}. Under an appropriate normalization of basis vectors these numbers are the *Racah coefficients* or *classical $6j$-symbols*. More generally, if $\mathcal{C} = U_{\mathsf{q}}(\mathfrak{sl}_2)$, where q is not a root of unity, then the numbers $(\Phi^{i_4}_{i_1 i_2 i_3})_{jl}$ are called q-*Racah coefficients* or *quantum $6j$-symbols* [CFS].

Further, the evaluation and coevaluation maps define elements

$$(1.42.6) \qquad \alpha_{ij} \in (H^j_{ii^*})^* \quad \text{and} \quad \beta_{ij} \in H^j_{ii^*}, \, j \in I_0.$$

Now the axioms of a rigid monoidal category, i.e., the triangle and pentagon identities and the rigidity axioms translate into non-linear algebraic equations with respect to the $6j$-symbols $(\Phi^{i_4}_{i_1 i_2 i_3})_{jl}$ and vectors α_{ij}, β_{ij}.

Exercise 1.42.7. Write down explicitly the relation on $6j$ symbols coming from the pentagon identity. If $\mathcal{C} = \mathrm{Rep}(\mathfrak{sl}_2(\mathbb{C}))$ this relation is called the *Elliott-Biedenharn* relation ([CFS]).

Proposition 1.42.4 gives rise to the following general problem of *categorification* of based rings which is one of the main problems in the structure theory of tensor categories.

Problem 1.42.8. Given a based ring R, describe (up to equivalence) all multitensor categories over k whose Grothendieck ring is isomorphic to R.

It is clear from the above explanations that this problem is equivalent to finding all solutions of the system of algebraic equations coming from the axioms of the rigid monoidal category modulo the group of automorphisms of the spaces H_{ij}^k ("gauge transformations"). In general, this problem is very difficult because the system of equations involved is nonlinear, contains many unknowns and is usually over-determined. In particular, it is not clear a priori whether for a given R this system has at least one solution, and if it does, whether the set of these solutions is finite. It is therefore amazing that the theory of tensor categories allows one to solve the categorification problem in a number of nontrivial cases. This will be done in later parts of these notes; now we will only mention the simplest result in this direction, which follows from the results of Subsection 1.7.

Let $\mathbb{Z}[G]$ be the group ring of a group G, with basis $\{g \in G\}$ and involution $g^* = g^{-1}$. Clearly, $\mathbb{Z}[G]$ is a unital based ring.

Proposition 1.42.9. *The categorifications of $\mathbb{Z}[G]$ are Vec_G^ω, and they are parametrized by $H^3(G, k^\times)/\mathrm{Out}(G)$.*

Remark 1.42.10. It is tempting to say that any $\mathbb{Z}_+$-ring R has a canonical categorification over any field k: one can take the skeletal semisimple category $\mathcal{C} = \mathcal{C}_R$ over k whose Grothendieck group is R, define the tensor product functor on $\mathcal{C}$ according to the multiplication in R, and then "define" the associativity isomorphism to be the identity (which appears to make sense because the category is skeletal, and therefore by the associativity of R one has $(X \otimes Y) \otimes Z = X \otimes (Y \otimes Z)$). However, a more careful consideration shows that this approach does not actually work. Namely, such "associativity isomorphism" fails to be functorial with respect to morphisms; in other words, if $g : Y \to Y$ is a morphism, then $(\mathrm{Id}_X \otimes g) \otimes \mathrm{Id}_Z$ is not always equal to $\mathrm{Id}_X \otimes (g \otimes \mathrm{Id}_Z)$.

To demonstrate this explicitly, denote the simple objects of the category $\mathcal{C}$ by X_i, $i = 1, ..., r$, and let $X_i \otimes X_j = \oplus_k N_{ij}^l X_l$. Take $X = X_i$, $Y = mX_j$, and $Z = X_l$; then g is an m by m matrix over k. The algebra $\mathrm{End}((X \otimes Y) \otimes Z) = \mathrm{End}(X \otimes (Y \otimes Z))$ is equal to $\oplus_s \mathrm{Mat}_{mn_s}(k)$, where

$$n_s = \sum_p N_{ij}^p N_{pl}^s = \sum_q N_{iq}^s N_{jl}^q,$$

and in this algebra we have

$$(\mathrm{Id}_X \otimes g) \otimes \mathrm{Id}_Z = \oplus_{p=1}^r \mathrm{Id}_{N_{ij}^p} \otimes g \otimes \mathrm{Id}_{N_{pl}^s},$$

$$\mathrm{Id}_X \otimes (g \otimes \mathrm{Id}_Z) = \oplus_{q=1}^r \mathrm{Id}_{N_{iq}^s} \otimes g \otimes \mathrm{Id}_{N_{jl}^q},$$

We see that these two matrices are, in general, different, which shows that the identity "associativity isomorphism" is not functorial.

1.43. **Semisimplicity of multifusion rings.**

Definition 1.43.1. A $*$-algebra is an associative algebra B over $\mathbb{C}$ with an antilinear anti-involution $* : B \to B$ and a linear functional $\tau : B \to \mathbb{C}$ such that $\tau(ab) = \tau(ba)$, and the Hermitian form $\tau(ab^*)$ is positive definite.

Obviously, any semisimple algebra $B = \oplus_{i=1}^r \mathrm{Mat}_i(\mathbb{C})$ is a $*$-algebra. Namely, if $p_i > 0$ are any positive numbers for $i = 1, ..., r$ then one can define $*$ to be the usual hermitian adjoint of matrices, and set $\tau(a_1, ..., a_r) = \sum_i p_i \mathrm{Tr}(a_i)$. Conversely, it is easy to see that any 8-algebra structure on a finite dimensional semisimple algebra has this form up to an isomorphism (and the numbers p_i are uniquely determined, as traces of central idempotents of B).

It turns out that this is the most general example of a finite dimensional $*$-algebra. Namely, we have

Proposition 1.43.2. *Any finite dimensional $*$-algebra B is semisimple.*

Proof. If $M \subset B$ is a subbimodule, and $M^\perp$ is the orthogonal complement of M under the form $\tau(ab^*)$, then $M^\perp$ is a subbimodule of B, and $M \cap M^\perp = 0$ because of the positivity of the form. So we have $B = M \oplus M^\perp$. Thus B is a semisimple B-bimodule, which implies the proposition. $\square$

Corollary 1.43.3. *If e is a nonzero idempotent in a finite dimensional $*$-algebra B then $\tau(e) > 0$.*

The following proposition is obvious.

Proposition 1.43.4. *Let A be a based ring. Then the algebra $A \otimes_{\mathbb{Z}} \mathbb{C}$ is canonically a $*$-algebra.*

Corollary 1.43.5. *Let A be a multifusion ring. Then the algebra $A \otimes_{\mathbb{Z}} \mathbb{C}$ is semsimiple.*

Corollary 1.43.6. *Let X be a basis element of a fusion ring A. Then there exists $n > 0$ such that $\tau(X^n) > 0$.*

Proof. Since $\tau(X^n(X^*)^n) > 0$ for all $n > 0$, X is not nilpotent. Let

$$q(x) := \prod_{i=0}^r (x - a_i)^{m_i}$$

be the minimal polynomial of X (a_i are distinct). Assume that $a_0 \neq 0$ (we can do so since X is not nilpotent). Let

$$g(t) = \prod_{i=1}^{r}(x - a_i)^{m_i}xh(x),$$

where h is a polynomial chosen in such a way that $g(a_0) = 1$, $g^{(j)}(a_0) = 0$ for $j = 1, ..., m_0 - 1$ (this is clearly possible). Then $g(X)$ is an idempotent, so by Corollary 1.43.3, $\tau(g(X)) > 0$. Hence there exists $n > 0$ such that $\tau(X^n) \neq 0$, as desired. $\square$

1.44. **The Frobenius-Perron theorem.** The following classical theorem from linear algebra [Ga, XIII.2] plays a crucial role in the theory of tensor categories.

Theorem 1.44.1. *Let B be a square matrix with nonnegative entries.*

(1) *B has a nonnegative real eigenvalue. The largest nonnegative real eigenvalue $\lambda(B)$ of B dominates the absolute values of all other eigenvalues μ of B: $|\mu| \leq \lambda(B)$ (in other words, the spectral radius of B is an eigenvalue). Moreover, there is an eigenvector of B with nonnegative entries and eigenvalue $\lambda(B)$.*

(2) *If B has strictly positive entries then $\lambda(B)$ is a simple positive eigenvalue, and the corresponding eigenvector can be normalized to have strictly positive entries. Moreover, $|\mu| < \lambda(B)$ for any other eigenvalue μ of B.*

(3) *If B has an eigenvector $\mathbf{v}$ with strictly positive entries, then the corresponding eigenvalue is $\lambda(B)$.*

Proof. Let B be an n by n matrix with nonnegative entries. Let us first show that B has a nonnegative eigenvalue. If B has an eigenvector $\mathbf{v}$ with nonnegative entries and eigenvalue 0, then there is nothing to prove. Otherwise, let Σ be the set of column vectors $\mathbf{x} \in \mathbb{R}^n$ with with nonnegative entries x_i and $s(\mathbf{x}) := \sum x_i$ equal to 1 (this is a simplex). Define a continuous map $f_B : \Sigma \to \Sigma$ by $f_B(\mathbf{x}) = \frac{B\mathbf{x}}{s(B\mathbf{x})}$. By the Brouwer fixed point theorem, this map has a fixed point $\mathbf{f}$. Then $B\mathbf{f} = \lambda\mathbf{f}$, where $\lambda > 0$. Thus the eigenvalue $\lambda(B)$ is well defined, and B always has a nonnegative eigenvector $\mathbf{f}$ with eigenvalue $\lambda = \lambda(B)$.

Now assume that B has strictly positive entries. Then $\mathbf{f}$ must have strictly positive entries f_i. If $\mathbf{d}$ is another real eigenvector of B with eigenvalue λ, let z be the smallest of the numbers of d_i/f_i. Then the vector $\mathbf{v} = \mathbf{d} - z\mathbf{f}$ satisfies $B\mathbf{v} = \lambda\mathbf{v}$, has nonnegative entries and has one entry equal to zero. Hence $\mathbf{v} = 0$ and λ is a simple eigenvalue.

Now let $\mathbf{y} = (y_1, ..., y_n) \in \mathbb{C}^n$ be a row vector. Define the norm $|\mathbf{y}| := \sum |y_j| f_j$. Then

$$|\mathbf{y}B| = \sum_j |\sum_i y_i b_{ij}| f_j \leq \sum_{i,j} |y_i| b_{ij} f_j = \lambda |\mathbf{y}|,$$

and the equality holds if and only if all the complex numbers $\sum y_i b_{ij}$ which are nonzero have the same argument. So if $\mathbf{y}B = \mu\mathbf{y}$, then $|\mu| \leq \lambda$, and if $|\mu| = \lambda$ then all y_i which are nonzero have the same argument, so we can renormalize $\mathbf{y}$ to have nonnegative entries. This implies that $\mu = \lambda$. Thus, part (2) is proved.

Now consider the general case (B has nonnegative entries). Assume that B has a row eigenevector $\mathbf{y}$ with strictly positive entries and eigenvalue μ. Then

$$\mu\mathbf{y}\mathbf{f} = \mathbf{y}B\mathbf{f} = \lambda\mathbf{y}\mathbf{f},$$

which implies $\mu = \lambda$, as $\mathbf{y}\mathbf{f} \neq 0$. This implies (3).

It remains to finish the proof of part (1) (i.e. to prove that $\lambda(B)$ dominates all other eigenvalues of B). Let Γ_B be the oriented graph whose vertices are labeled by $1, ..., n$, and there is an edge from j to i if and only if $b_{ij} > 0$. Let us say that i is accessible from j if there is a path in Γ_B leading from j to i. Let us call B irreducible if any vertex is accessible from any other. By conjugating B by a permutation matrix if necessary, we can get to a situation when $i \geq j$ implies that i is accessible from j. This means that B is a block upper triangular matrix, whose diagonal blocks are irreducible. So it suffices to prove the statement in question for irreducible B.

But if B is irreducible, then for some N the matrix $1 + B + \cdots + B^N$ has strictly positive entries. So the nonnegative eigenvector $\mathbf{f}$ of B with eigenvalue $\lambda(B)$ is actually strictly positive, and one can run the argument in the proof of part (2) with a norm bound (all that is used in this argument is the positivity of $\mathbf{f}$). Hence, the result follows from (2). $\qquad\square$

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