1.32. **The Andruskiewitsch-Schneider conjecture.** It is easy to see that any Hopf algebra generated by grouplike and skew-primitive elements is automatically pointed.

On the other hand, there exist pointed Hopf algebras which are not generated by grouplike and skew-primitive elements. Perhaps the simplest example of such a Hopf algebra is the algebra of regular functions on the Heisenberg group (i.e. the group of upper triangular 3 by 3 matrices with ones on the diagonal). It is easy to see that the commutative Hopf algebra $H$ is the polynomial algebra in generators $x, y, z$ (entries of the matrix), so that $x, y$ are primitive, and

$$\Delta(z) = z \otimes 1 + 1 \otimes z + x \otimes y.$$

Since the only grouplike element in $H$ is 1, and the only skew-primitive elements are $x, y$, $H$ is not generated by grouplike and skew-primitive elements.

However, one has the following conjecture, due to Andruskiewitsch and Schneider.

**Conjecture 1.32.1.** Any finite dimensional pointed Hopf algebra over a field of characteristic zero is generated in degree 1 of its coradical filtration, i.e., by grouplike and skew-primitive elements.

It is easy to see that it is enough to prove this conjecture for coradically graded Hopf algebras; this has been done in many special cases (see [AS]).

The reason we discuss this conjecture here is that it is essentially a categorical statement. Let us make the following definition.

**Definition 1.32.2.** We say that a tensor category $\mathcal{C}$ is *tensor-generated* by a collection of objects $X_\alpha$ if every object of $\mathcal{C}$ is a subquotient of a finite direct sum of tensor products of $X_\alpha$.

**Proposition 1.32.3.** *A pointed Hopf algebra $H$ is generated by grouplike and skew-primitive elements if and only if the tensor category $H - \mathrm{comod}$ is tensor-generated by objects of length 2.*

*Proof.* This follows from the fact that matrix elements of the tensor product of comodules $V, W$ for $H$ are products of matrix elements of $V, W$. □

Thus, one may generalize Conjecture 1.32.1 to the following conjecture about tensor categories.

**Conjecture 1.32.4.** Any finite pointed tensor category over a field of characteristic zero is tensor generated by objects of length 2.

As we have seen, this property fails for infinite categories, e.g., for the category of rational representations of the Heisenberg group. In fact, this is very easy to see categorically: the center of the Heisenberg group acts trivially on 2-dimensional representations, but it is not true for a general rational representation.

### 1.33. **The Cartier-Kostant theorem.**

**Theorem 1.33.1.** *Any cocommutative Hopf algebra $H$ over an algebraically closed field of characteristic zero is of the form $k[G] \ltimes U(\mathfrak{g})$, where $\mathfrak{g}$ is a Lie algebra, and $G$ is a group acting on $\mathfrak{g}$.*

*Proof.* Let $G$ be the group of grouplike elements of $H$. Since $H$ is cocommutative, it is pointed, and $\mathrm{Ext}^1(g,h) = 0$ if $g, h \in G$, $g \neq h$. Hence the category $\mathcal{C} = H-$comod splits into a direct sum of blocks $\mathcal{C} = \oplus_{g\in G}\mathcal{C}_g$, where $\mathcal{C}_g$ is the category of objects of $\mathcal{C}$ which have a filtration with successive quotients isomorphic to $g$. So $H = \oplus_{g\in G} H_g$, where $\mathcal{C}_g = H_g-$comod, and $H_g = gH_1$. Moreover, $A = H_1$ is a Hopf algebra, and we have an action of $G$ on $A$ by Hopf algebra automorphisms.

Now let $\mathfrak{g} = \mathrm{Prim}(A) = \mathrm{Prim}(H)$. This is a Lie algebra, and the group $G$ acts on it (by conjugation) by Lie algebra automorphisms. So we need just to show that the natural homomorphism $\psi : U(\mathfrak{g}) \to A$ is actually an isomorphism.

It is clear that any morphism of coalgebras preserves the coradical filtration, so we can pass to the associated graded morphism $\psi_0 : S\mathfrak{g} \to A_0$, where $A_0 = \mathrm{gr}(A)$. It is enough to check that $\psi_0$ is an isomorphism.

The morphism $\psi_0$ is an isomorphism in degrees 0 and 1, and by Corollary 1.29.7, it is injective. So we only need to show surjectivity.

We prove the surjectivity in each degree $n$ by induction. To simplify notation, let us identify $S\mathfrak{g}$ with its image under $\psi_0$. Suppose that the surjectivity is known in all degrees below $n$. Let $z$ be a homogeneous element in $A_0$ of degree $n$. Then it is easy to see from the counit axiom that

$$\Delta(z) - z\otimes 1 - 1\otimes z = u \tag{1.33.1}$$

where $u \in S\mathfrak{g}\otimes S\mathfrak{g}$ is a symmetric element (as $\Delta$ is cocommutative).

Equation 1.33.1 implies that the element $u$ satisfies the equation

$$(\Delta\otimes \mathrm{Id})(u) + u\otimes 1 = (\mathrm{Id}\otimes\Delta)(u) + 1\otimes u. \tag{1.33.2}$$

**Lemma 1.33.2.** *Let $V$ be a vector space over a field $k$ of characteristic zero. Let $u \in SV\otimes SV$ be a symmetric element satisfying equation (1.33.2). Then $u = \Delta(w) - w\otimes 1 - 1\otimes w$ for some $w \in SV$.*

*Proof.* Clearly, we may assume that $V$ is finite dimensional. Regard $u$ as a polynomial function on $V^* \times V^*$; our job is to show that

$$u(x,y) = w(x+y) - w(x) - w(y)$$

for some polynomial $w$.

If we regard $u$ as a polynomial, equation (1.33.2) takes the form of the 2-cocycle condition

$$u(x+y,t) + u(x,y) = u(x,y+t) + u(y,t).$$

Thus $u$ defines a group law on $U := V^* \oplus k$, given by

$$(x,a) + (y,b) = (x+y, a+b+u(x,y)).$$

Clearly, we may assume that $u$ is homogeneous, of some degree $d \neq 1$. Since $u$ is symmetric, the group $U$ is abelian. So in $U$ we have

$$((x,0)+(x,0)) + ((y,0)+(y,0)) = ((x,0)+(y,0)) + ((x,0)+(y,0))$$

Computing the second component of both sides, we get

$$u(x,x) + u(y,y) + 2^d u(x,y) = 2u(x,y) + u(x+y,x+y).$$

So one can take $w(x) = (2^d-2)^{-1}u(x,x)$, as desired. □

Now, applying Lemma 1.33.2, we get that there exists $w \in A_0$ such that $z - w$ is a primitive element, which implies that $z - w \in A_0$, so $z \in A_0$. □

**Remark 1.33.3.** The Cartier-Kostant theorem implies that any cocommutative Hopf algebra over an algebraically closed field of characteristic zero in which the only grouplike element is 1 is of the form $U(\mathfrak{g})$, where $\mathfrak{g}$ is a Lie algebra (a version of the Milnor-Moore theorem), in particular is generated by primitive elements. The latter statement is false in positive charactersitic. Namely, consider the commutative Hopf algebra $\mathbb{Q}[x,z]$ where $x, z$ are primitive, and set $y = z + x^p/p$, where $p$ is a prime. Then

$$\Delta(y) = y \otimes 1 + 1 \otimes y + \sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i} x^i \otimes x^{p-i}. \tag{1.33.3}$$

Since the numbers $\frac{1}{p}\binom{p}{i}$ are integers, this formula (together with $\Delta(x) = x \otimes 1 + 1 \otimes x$, $S(x) = -x$, $S(y) = -y$) defines a Hopf algebra structure on $H = k[x,y]$ for any field $k$, in particular, one of characteristic $p$. But if $k$ has characteristic $p$, then it is easy to see that $H$ is not generated by primitive elements (namely, the element $y$ is not in the subalgebra generated by them).

The Cartier-Kostant theorem implies that any affine pro-algebraic group scheme over a field of characteristic zero is in fact a pro-algebraic group. Namely, we have

**Corollary 1.33.4.** *Let $H$ be a commutative Hopf algebra over a field $k$ of characteristic zero. Then $H$ has no nonzero nilpotent elements.*

*Proof.* It is clear that $H$ is a union of finitely generated Hopf subalgebras (generated by finite dimensional subcoalgebras of $H$), so we may assume that $H$ is finitely generated. Let $\mathfrak{m}$ be the kernel of the counit of $H$, and $B = \cup_{n=1}^{\infty}(H/\mathfrak{m}^n)^*$ (i.e., $B$ is the continuous dual of the formal completion of $H$ near the ideal $\mathfrak{m}$). It is easy to see that $B$ is a cocommutative Hopf algebra, and its only grouplike element is 1. So by the Cartier-Kostant theorem $B = U(\mathfrak{g})$, where $\mathfrak{g} = (\mathfrak{m}/\mathfrak{m}^2)^*$. This implies that $G = \mathrm{Spec}(H)$ is smooth at $1 \in G$, i.e. it is an algebraic group, as desired. $\square$

**Remark 1.33.5.** Note that Corollary 1.33.4 is a generalization of Corollary 1.27.6.

1.34. **Quasi-bialgebras.** Let us now discuss reconstruction theory for quasi-fiber functors. This leads to the notion of quasi-bialgebras and quasi-Hopf algebras, which were introduced by Drinfeld in [Dr1] as linear algebraic counterparts of abelian monoidal categories with quasi-fiber functors.

**Definition 1.34.1.** Let $\mathcal{C}$ be an abelian monoidal category over $k$, and $(F, J) : \mathcal{C} \to \mathrm{Vec}$ be a quasi-fiber functor. $(F, J)$ is said to be *normalized* if $J_{\mathbf{1}X} = J_{X\mathbf{1}} = \mathrm{Id}_{F(X)}$ for all $X \in \mathcal{C}$.

**Definition 1.34.2.** Two quasi-fiber functors $(F, J_1)$ and $(F, J_2)$ are said to be *twist equivalent* (by the twist $J_1^{-1}J_2$).

Since for a quasi-fiber functor (unlike a fiber functor), the isomorphism $J$ is not required to satisfy any equations, it typically does not carry any valuable structural information, and thus it is more reasonable to classify quasi-fiber functors not up to isomorphism, but rather up to twist equivalence combined with isomorphism.

**Remark 1.34.3.** It is easy to show that any quasi-fiber functor is equivalent to a normalized one.

Now let $\mathcal{C}$ be a finite abelian monoidal category over $k$, and let $(F, J)$ be a normalized quasi-fiber functor. Let $H = \mathsf{End}\, F$ be the corresponding finite dimensional algebra. Then $H$ has a coproduct $\Delta$ and a counit $\varepsilon$ defined exactly as in the case of a fiber functor, which are algebra homomorphisms. The only difference is that, in general,

$\Delta$ is not coassociative, since $J$ does not satisfy the monoidal structure axiom. Rather, there is an invertible element $\Phi \in H^{\otimes 3}$, defined by the commutative diagram

(1.34.1)

$$\begin{tikzcd}
(F(X) \otimes F(Y)) \otimes F(Z) \arrow[r, "\Phi_{F(X),F(Y),F(Z)}"] \arrow[d, "J_{X,Y}\otimes \mathrm{Id}_{F(Z)}"'] & F(X) \otimes (F(Y) \otimes F(Z)) \arrow[d, "\mathrm{Id}_{F(X)}\otimes J_{Y,Z}"'] \\
F(X \otimes Y) \otimes F(Z) \arrow[d, "J_{X\otimes Y,Z}"'] & F(X) \otimes F(Y \otimes Z) \arrow[d, "J_{X,Y\otimes Z}"'] \\
F((X \otimes Y) \otimes Z) \arrow[r, "F(a_{X,Y,Z})"] & F(X \otimes (Y \otimes Z))
\end{tikzcd}$$

for all $X, Y, Z \in \mathcal{C}$, and we have the following proposition.

**Proposition 1.34.4.** *The following identities hold:*

$$(\mathrm{Id} \otimes \Delta)(\Delta(h)) = \Phi(\Delta \otimes \mathrm{Id})(\Delta(h))\Phi^{-1}, \quad h \in H, \tag{1.34.2}$$

(1.34.3)

$$(\mathrm{Id} \otimes \mathrm{Id} \otimes \Delta)(\Phi)(\Delta \otimes \mathrm{Id} \otimes \mathrm{Id})(\Phi) = (1 \otimes \Phi)(\mathrm{Id} \otimes \Delta \otimes \mathrm{Id})(\Phi)(\Phi \otimes 1),$$

$$(\varepsilon \otimes \mathrm{Id})(\Delta(h)) = h = (\mathrm{Id} \otimes \varepsilon)(\Delta(h)), \tag{1.34.4}$$

$$(\mathrm{Id} \otimes \varepsilon \otimes \mathrm{Id})(\Phi) = 1 \otimes 1. \tag{1.34.5}$$

*Proof.* The first identity follows from the definition of $\Phi$, the second one from the pentagon axiom for $\mathcal{C}$, the third one from the condition that $(F, J)$ is normalized, and the fourth one from the triangle axiom and the condition that $(F, J)$ is normalized. $\square$

**Definition 1.34.5.** An associative unital $k$-algebra $H$ equipped with unital algebra homomorphisms $\Delta : H \to H \otimes H$ (the coproduct) and $\varepsilon : H \to k$ (the counit) and an invertible element $\Phi \in H^{\otimes 3}$ satisfying the identities of Proposition 1.34.4 is called a *quasi-bialgebra*. The element $\Phi$ is called the *associator* of $H$.

Thus, the notion of a quasi-bialgebra is a generalization of the notion of a bialgebra; namely, a bialgebra is a quasi-bialgebra with $\Phi = 1$.[11]

For a quasi-bialgebra $H$, the tensor product of (left) $H$-modules $V$ and $W$ is an $H$-module via $\Delta$, i.e., in the same way as for bialgebras. Also, it follows from (1.34.2) that for any $H$-modules $U, V, W$ the mapping

$$a_{U,V,W} : (U \otimes V) \otimes W \cong U \otimes (V \otimes W) : u \otimes v \otimes w \mapsto \Phi(u \otimes v \otimes w) \tag{1.34.6}$$

[11]However, note that $\Delta$ can be coassociative even if $\Phi \neq 1$.

is an $H$-module isomorphism. The axiom (1.34.4) implies that the natural maps $l_V = \mathrm{Id} : \mathbf{1} \otimes V \xrightarrow{\sim} V$ and $r_V = \mathrm{Id} : V \otimes \mathbf{1} \xrightarrow{\sim} V$ are also $H$-module isomorphisms. Finally, equations (1.34.3) and (1.34.5) say, respectively, that the pentagon axiom (1.1.2) and the triangle axiom (1.2.1) are satisfied for $\mathbf{Rep}(H)$. In other words, $\mathbf{Rep}(H)$ is a monoidal category.

**Definition 1.34.6.** A *twist* for a quasi-bialgebra $H$ is an invertible element $J \in H \otimes H$ such that $(\varepsilon \otimes \mathrm{Id})(J) = (\mathrm{Id} \otimes \varepsilon)(J) = 1$. Given a twist, we can define a new quasi-bialgebra $H^J$ which is $H$ as an algebra, with the same counit, the coproduct given by

$$\Delta^J(x) = J^{-1}\Delta(x)J,$$

and the associator given by

$$\Phi^J = (\mathrm{Id} \otimes J)^{-1}(\mathrm{Id} \otimes \Delta)(J)^{-1}\Phi(\Delta \otimes \mathrm{Id})(J)(J \otimes \mathrm{Id})$$

The algebra $H^J$ is called *twist equivalent* to $H$, by the twist $J$.

It is easy to see that twist equivalent quasi-fiber functors produce twist-equivalent quasi-bialgebras, and vice versa. Also, we have the following proposition.

**Proposition 1.34.7.** *If a finite $k$-linear abelian monoidal category $\mathcal{C}$ admits a quasi-fiber functor, then this functor is unique up to twisting.*

*Proof.* Let $X_i$, $i = 1, ..., n$ be the simple objects of $\mathcal{C}$. The functor $F$ is exact, so it is determined up to isomorphism by the numbers $d_i = \dim F(X_i)$. So our job is to show that these numbers are uniquely determined by $\mathcal{C}$.

Let $N_i = (N_{ij}^k)$ be the matrix of left multiplication by $X_i$ in the Grothendieck ring of $\mathcal{C}$ in the basis $\{X_j\}$, i.e.

$$X_iX_j = \sum N_{ij}^k X_k$$

(so, $k$ labels the rows and $j$ labels the columns of $N_i$).

We claim that $d_i$ is the spectral radius of $N_i$. Indeed, on the one hand, we have

$$\sum N_{ij}^m d_m = d_i d_j,$$

so $d_i$ is an eigenvalue of $N_i^T$, hence of $N_i$. On the other hand, if $e_j$ is the standard basis of $\mathbb{Z}^n$ then for any $r \geq 0$ the sum of the coordinates of the vector $N_i^r e_j$ is the length of the object $X_i^{\otimes r} \otimes X_j$, so it is dominated by $d_i^r d_j$. This implies that the spectral radius of $N_i$ is at most $d_i$. This means that the spectral radius is exactly $d_i$, as desired. □

Therefore, we have the following reconstruction theorem.

**Theorem 1.34.8.** *The assignments* $(\mathcal{C}, F) \mapsto H = \mathsf{End}(F)$*,* $H \mapsto (\mathrm{Rep}(H), \mathrm{Forget})$ *are mutually inverse bijections between*

*1) finite* $k$*-linear abelian monoidal categories* $\mathcal{C}$ *admitting a quasi-fiber functor, up to monoidal equivalence of categories.*

*2) finite dimensional quasi-bialgebras* $H$ *over* $k$ *up to twist equivalence and isomorphism.*

*Proof.* Straightforward from the above. □

**Exercise 1.34.9.** Suppose that in the situation of Exercise 1.21.6, the functor $F$ is equipped with a quasi-monoidal structure $J$, i.e. an isomorphism $J : F(\bullet) \otimes F(\bullet) \to F(\bullet \otimes \bullet)$, such that $J_{\mathbf{1}X} = J_{X\mathbf{1}} = \mathrm{Id}_{F(X)}$. Show that this endows $H$ with the structure of a quasi-bialgebra, such that $(F, J)$ defines a monoidal equivalence $\mathcal{C} \to \mathbf{Rep}(H)$.

**Remark 1.34.10.** Proposition 1.34.7 is false for infinite categories. For example, it is known that if $\mathcal{C} = \mathrm{Rep}(SL_2(\mathbb{C}))$, and $V \in \mathcal{C}$ is a 2-dimensional repesentation, then there exists a for any positive integer $n \geq 2$ there exists a fiber functor on $\mathcal{C}$ with $\dim F(V) = n$ (see [Bi]).

1.35. **Quasi-bialgebras with an antipode and quasi-Hopf algebras.** Now consider the situation of the previous subsection, and assume that the category $\mathcal{C}$ has right duals. In this case, by Proposition 1.13.5, the right dualization functor is exact; it is also faithful by Proposition 1.10.9. Therefore, the functor $F(V^*)^*$ is another quasi-fiber functor on $\mathcal{C}$. So by Proposition 1.34.7, this functor is isomorphic to $F$. Let us fix such an isomorphism $\xi = (\xi_V)$, $\xi_V : F(V) \to F(V^*)^*$. Then we have natural linear maps $k \to F(V) \otimes F(V^*)$, $F(V^*) \otimes F(V) \to k$ constructed as in Exercise 1.10.6, which can be regarded as linear maps $\hat{\alpha} : F(V) \to F(V^*)^*$ and $\hat{\beta} : F(V^*)^* \to F(V)$. Thus, the quasi-bialgebra $H = \mathsf{End}(F)$ has the following additional structures.

1. The elements $\alpha, \beta \in H$ such that for any $V \in \mathcal{C}$, $\alpha_V = \xi_V^{-1} \circ \hat{\alpha}_V$, $\beta_V = \hat{\beta}_V \circ \xi_V$. Note that $\alpha$ and $\beta$ are not necessarily invertible.

2. The antipode $S : H \to H$, which is a unital algebra antihomomorphism such that if $\Delta(a) = \sum_i a_i^1 \otimes a_i^2$, $a \in H$, then

$$\sum_i S(a_i^1)\alpha a_i^2 = \varepsilon(a)\alpha, \qquad \sum_i a_i^1 \beta S(a_i^2) = \varepsilon(a)\beta. \tag{1.35.1}$$

Namely, for $a \in H$ $S(a)$ acts on $F(V)$ by $\xi^{-1} \circ a^*_{F(V^*)} \circ \xi$.

Let us write the associator as $\Phi = \sum_i \Phi_i^1 \otimes \Phi_i^2 \otimes \Phi_i^3$ and its inverse as $\sum \bar{\Phi}_i^1 \otimes \bar{\Phi}_i^2 \otimes \bar{\Phi}_i^3$.

**Proposition 1.35.1.** *One has*

$$\sum \Phi_i^1 \beta S(\Phi_i^2) \alpha \Phi_i^3 = 1, \qquad \sum S(\bar{\Phi}_i^1) \alpha \bar{\Phi}_i^2 \beta S(\bar{\Phi}_i^3) = 1. \tag{1.35.2}$$

*Proof.* This follows directly from the duality axioms. $\square$

**Definition 1.35.2.** An *antipode* on a quasi-bialgebra $H$ is a triple $(S, \alpha, \beta)$, where $S : H \to H$ is a unital antihomomorphism and $\alpha, \beta \in H$, satisfying identities (1.35.1) and (1.35.2).

A *quasi-Hopf algebra* is a quasi-bialgebra $(H, \Delta, \varepsilon, \Phi)$ for which there exists an antipode $(S, \alpha, \beta)$ such that $S$ is bijective.

Thus, the notion of a quasi-Hopf algebra is a generalization of the notion of a Hopf algebra; namely, a Hopf algebra is a quasi-Hopf algebra with $\Phi = 1$, $\alpha = \beta = 1$.

We see that if in the above setting $\mathcal{C}$ has right duals, then $H = \mathsf{End}(F)$ is a finite dimensional bialgebra admitting antipode, and if $\mathcal{C}$ is rigid (i.e., a tensor category), then $H$ is a quasi-Hopf algebra.

Conversely, if $H$ is a quasi-bialgebra with an antipode, then the category $\mathcal{C} = \mathrm{Rep}(H)$ admits right duals. Indeed, the right dual module of an $H$-module $V$ is defined as in the Hopf algebra case: it is the dual vector space $V^*$ with the action of $H$ given by

$$\langle h\phi, v \rangle = \langle \phi, S(h)v \rangle, \qquad v \in V, \phi \in V^*, h \in H.$$

Let $\sum v_i \otimes f_i$ be the image of $\mathrm{Id}_V$ under the canonical isomorphism $\mathsf{End}(V) \xrightarrow{\sim} V \otimes V^*$. Then the evaluation and coevaluation maps are defined using the elements $\alpha$ and $\beta$:

$$\mathrm{ev}_V(f \otimes v) = f(\alpha v), \mathrm{coev}_V(1) = \sum \beta v_i \otimes f_i.$$

Axiom (1.35.1) is then equivalent to $\mathrm{ev}_V$ and $\mathrm{coev}_V$ being $H$-module maps. Equations (1.35.2) are equivalent, respectively, to axioms (1.10.1) and (1.10.2) of a right dual.

If $S$ is invertible, then the right dualization functor is an equivalence of categories, so the representation category $\mathrm{Rep}(H)$ of a quasi-Hopf algebra $H$ is rigid, i.e., a tensor category.

**Exercise 1.35.3.** Let $H := (H, \Delta, \varepsilon, \Phi, S, \alpha, \beta)$ be a quasi-bialgebra with an antipode, and $u \in H$ be an invertible element.

(i) Show that if one sets

$$\overline{S}(h) = uS(h)u^{-1}, \qquad \overline{\alpha} = u\alpha, \quad \text{and} \quad \overline{\beta} = \beta u^{-1} \tag{1.35.3}$$

then the triple $(\overline{S}, \overline{\alpha}, \overline{\beta})$ is an antipode.

(ii) Conversely, show that any $\overline{S}$, $\overline{\alpha}$, and $\overline{\beta}$ satisfying conditions (1.35.1) and (1.35.2) are given by formulas (1.35.3) for a uniquely defined $u$.

Hint. If $H$ is finite dimensional, (ii) can be formally deduced from the uniqueness of the right dual in a tensor category up to a unique isomorphism. Use this approach to obtain the unique possible formula for $u$, and check that it does the job for any $H$.

**Remark 1.35.4.** The non-uniqueness of $S$, $\alpha$, and $\beta$ observed in Exercise 1.35.3 reflects the freedom in choosing the isomorphism $\xi$.

**Example 1.35.5.** (cf. Example 1.10.14) Let $G$ be a finite group and let $\omega \in Z^3(G, k^\times)$ be a normalized 3-cocycle, see (1.3.1). Consider the algebra $H = \mathrm{Fun}(G, k)$ of $k$-valued functions on $G$ with the usual coproduct and counit. Set

$$\Phi = \sum \omega(f, g, h) p_f \otimes p_g \otimes p_h, \quad \alpha = \sum \omega(g, g^{-1}, g) p_g, \quad \beta = 1,$$

where $p_g$ is the primitive idempotent of $H$ corresponding to $g \in G$. It is straightforward to check that these data define a commutative quasi-Hopf algebra, which we denote $\mathrm{Fun}(G, k)_\omega$. The tensor category $\mathrm{Rep}(\mathrm{Fun}(G, k)_\omega)$ is obviously equivalent to $\mathrm{Vec}_G^\omega$.

It is easy to show that a twist of a quasi-bialgebra $H$ with an antipode is again a quasi-bialgebra with an antipode (this reflects the fact that in the finite dimensional case, the existence of an antipode for $H$ is the property of the category of finite dimensional representations of $H$). Indeed, if the twist $J$ and its inverse have the form

$$J = \sum_i a_i \otimes b_i, \ J^{-1} = \sum_i a_i' \otimes b_i'$$

then $H^J$ has an antipode $(S^J, \alpha^J, \beta^J)$ with $S^J = S$ and $\alpha^J = \sum_i S(a_i)\alpha b_i$, $\beta^J = \sum_i a_i' \beta S(b_i')$. Thus, we have the following reconstruction theorem.

**Theorem 1.35.6.** *The assignments* $(\mathcal{C}, F) \mapsto H = \mathsf{End}(F)$, $H \mapsto (\mathrm{Rep}(H), \mathrm{Forget})$ *are mutually inverse bijections between*

*(i) finite abelian $k$-linear monoidal categories $\mathcal{C}$ with right duals admitting a quasi-fiber functor, up to monoidal equivalence of categories, and finite dimensional quasi-bialgebras $H$ over $k$ with an antipode, up to twist equivalence and isomorphism;*

*(ii) finite tensor categories $\mathcal{C}$ admitting a quasi-fiber functor, up to monoidal equivalence of categories, and finite dimensional quasi-Hopf algebras $H$ over $k$, up to twist equivalence and isomorphism.*

**Remark 1.35.7.** One can define the dual notions of a coquasi-bialgebra and coquasi-Hopf algebra, and prove the corresponding reconstruction theorems for tensor categories which are not necessarily finite. This is straightforward, but fairly tedious, and we will not do it here.

1.36. **Twists for bialgebras and Hopf algebras.** Let $H$ be a bialgebra. We can regard it as a quasi-bialgebra with $\Phi = 1$. Let $J$ be a twist for $H$.

**Definition 1.36.1.** $J$ is called a bialgebra twist if $H^J$ is a bialgebra, i.e. $\Phi^J = 1$.

Thus, a bialgebra twist for $H$ is an invertible element $J \in H \otimes H$ such that $(\varepsilon \otimes \mathrm{Id})(J) = (\mathrm{Id} \otimes \varepsilon)(J) = 1$, and $J$ satisfies the *twist equation*

$$(\mathrm{Id} \otimes \Delta)(J)(\mathrm{Id} \otimes J) = (\Delta \otimes \mathrm{Id})(J)(J \otimes \mathrm{Id}). \tag{1.36.1}$$

**Exercise 1.36.2.** Show that if a bialgebra $H$ has an antipode $S$, and $J$ is a bialgebra twist for $H$, then the bialgebra $H^J$ also has an antipode. Namely, let $J = \sum a_i \otimes b_i$, $J^{-1} = \sum a_i' \otimes b_i'$, and set $Q_J = \sum_i S(a_i)b_i$. Then $Q_J$ is invertible with $Q_J^{-1} = \sum_i a_i' S(b_i')$, and the antipode of $H^J$ is defined by $S^J(x) = Q_J^{-1} S(x) Q_J$. In particular, a bialgebra twist of a Hopf algebra is again a Hopf algebra.

**Remark 1.36.3.** Twisting does not change the category of $H$-modules as a monoidal category, and the existence of an antipode (for finite dimensional $H$) is a categorical property (existence of right duals). This yields the above formulas, and then one easily checks that they work for any $H$.

Any twist on a bialgebra $H$ defines a fiber functor $(\mathrm{Id}, J)$ on the category $\mathrm{Rep}(H)$. However, two different twists $J_1, J_2$ may define isomorphic fiber functors. It is easy to see that this happens if there is an invertible element $v \in H$ such that

$$J_2 = \Delta(v) J_1 (v^{-1} \otimes v^{-1}).$$

In this case the twists $J_1$ and $J_2$ are called *gauge equivalent* by the gauge transformation $v$, and the bialgebras $H^{J_1}, H^{J_2}$ are isomorphic (by conjugation by $v$). So, we have the following result.

**Proposition 1.36.4.** *Let $H$ be a finite dimensional bialgebra. Then $J \mapsto (\mathrm{Id}, J)$ is a bijection between:*
*1) gauge equivalence classes of bialgebra twists for $H$, and*
*2) fiber functors on $\mathcal{C} = \mathrm{Rep}(H)$, up to isomorphism.*

*Proof.* By Proposition 1.34.7, any fiber functor on $\mathcal{C}$ is isomorphic to the forgetful functor $F$ as an additive functor. So any fiber functor, up to an isomorphism, has the form $(F, J)$, where $J$ is a bialgebra twist. Now it remains to determine when $(F, J_1)$ and $(F, J_2)$ are isomorphic. Let $v : (F, J_1) \to (F, J_2)$ be an isomorphism. Then $v \in H$ is an invertible element, and it defines a gauge transformation mapping $J_1$ to $J_2$. $\square$

**Proposition 1.36.5.** *Let $G$ be a group. Then fiber functors on* $\mathrm{Vec}_G$ *up to an isomorphism bijectively correspond to $H^2(G, k^\times)$.*

*Proof.* A monoidal structure on the forgetful functor $F$ is given by a function $J(g,h) : \delta_g \otimes \delta_h \to \delta_g \otimes \delta_h$, $J(g,h) \in k^\times$. It is easy to see that the monoidal structure condition is the condition that $J$ is a 2-cocycle, and two 2-cocycles define isomorphic monoidal structures if and only if they differ by a coboundary. Thus, equivalence classes of monoidal structures on $F$ are parametrized by $H^2(G, k^\times)$, as desired. □

**Remark 1.36.6.** Proposition 1.36.5 shows that there may exist non-isomorphic fiber functors on a given finite tensor category $\mathcal{C}$ defining isomorphic Hopf algebras. Indeed, all fiber functors on $\mathrm{Vec}_G$ yield the same Hopf algebra $\mathrm{Fun}(G, k)$. These fiber functors are, however, all equivalent to each other by monoidal autoequivalences of $\mathcal{C}$.

**Remark 1.36.7.** Since $\mathrm{Vec}_G^\omega$ does not admit fiber functors for cohomologically nontrivial $\omega$, Proposition 1.36.5 in fact classifies fiber functors on all categories $\mathrm{Vec}_G^\omega$.