1.21. **Bialgebras.** Let $\mathcal{C}$ be a finite monoidal category, and $(F, J) : \mathcal{C} \to \mathrm{Vec}$ be a fiber functor. Consider the algebra $H := \mathsf{End}(F)$. This algebra has two additional structures: the comultiplication $\Delta : H \to H \otimes H$ and the counit $\varepsilon : H \to k$. Namely, the comultiplication is defined by the formula

$$\Delta(a) = \alpha_{F,F}^{-1}(\widetilde{\Delta}(a)),$$

where $\widetilde{\Delta}(a) \in \mathsf{End}(F \otimes F)$ is given by

$$\widetilde{\Delta}(a)_{X,Y} = J_{X,Y}^{-1} a_{X \otimes Y} J_{X,Y},$$

and the counit is defined by the formula

$$\varepsilon(a) = a_{\mathbf{1}} \in k.$$

Theorem 1.21.1. *(i) The algebra H is a coalgebra with comultiplication Δ and counit ε.*

(ii) The maps Δ and ε are unital algebra homomorphisms.

Proof. The coassociativity of Δ follows form axiom (1.4.1) of a monoidal functor. The counit axiom follows from (1.4.3) and (1.4.4). Finally, observe that for all $\eta, \nu \in \mathsf{End}(F)$ the images under $\alpha_{F,F}$ of both $\Delta(\eta)\Delta(\nu)$ and $\Delta(\eta\nu)$ have components $J_{X,Y}^{-1}(\eta\nu)_{X \otimes Y} J_{X,Y}$; hence, Δ is an algebra homomorphism (which is obviously unital). The fact that ε is a unital algebra homomorphism is clear. $\square$

Definition 1.21.2. An algebra H equipped with a comultiplication Δ and a counit ε satisfying properties (i),(ii) of Theorem 1.21.1 is called a *bialgebra*.

Thus, Theorem 1.21.1 claims that the algebra $H = \mathsf{End}(F)$ has a natural structure of a bialgebra.

Now let H be any bialgebra (not necessarily finite dimensional). Then the category $\mathbf{Rep}(H)$ of representations (i.e., left modules) of H and its subcategory $\mathrm{Rep}(H)$ of finite dimensional representations of H are naturally monoidal categories (and the same applies to right modules). Indeed, one can define the tensor product of two H-modules X, Y to be the usual tensor product of vector spaces $X \otimes Y$, with the action of H defined by the formula

$$\rho_{X \otimes Y}(a) = (\rho_X \otimes \rho_Y)(\Delta(a)), \ a \in H$$

(where $\rho_X : H \to \mathsf{End}(X), \rho_Y : H \to \mathsf{End}(Y)$), the associativity isomorphism to be the obvious one, and the unit object to be the 1-dimensional space k with the action of H given by the counit, $a \to \varepsilon(a)$. Moreover, the forgetful functor $\mathrm{Forget} : \mathrm{Rep}(H) \to \mathrm{Vec}$ is a fiber functor.

Thus we see that one has the following theorem.

Theorem 1.21.3. *The assignments* $(\mathcal{C}, F) \mapsto H = \mathsf{End}(F)$*,* $H \mapsto (\mathrm{Rep}(H), \mathrm{Forget})$ *are mutually inverse bijections between*

1) finite abelian k*-linear monoidal categories* $\mathcal{C}$ *with a fiber functor* F*, up to monoidal equivalence and isomorphism of monoidal functors;*

2) finite dimensional bialgebras H *over* k *up to isomorphism.*

Proof. Straightforward from the above. □

Theorem 1.21.3 is called *the reconstruction theorem for finite dimensional bialgebras* (as it reconstructs the bialgebra H from the category of its modules using a fiber functor).

Exercise 1.21.4. Show that the axioms of a bialgebra are self-dual in the following sense: if H is a finite dimensional bialgebra with multiplication $\mu : H \otimes H \to H$, unit $i : k \to H$, comultiplication $\Delta : H \to H \otimes H$ and counit $\varepsilon : H \to k$, then H^* is also a bialgebra, with the multiplication Δ^*, unit ε^*, comultiplication μ^*, and counit i^*.

Exercise 1.21.5. (i) Let G be a finite monoid, and $\mathcal{C} = \mathrm{Vec}_G$. Let $F : \mathcal{C} \to \mathrm{Vec}$ be the forgetful functor. Show that $H = \mathsf{End}(F)$ is the bialgebra $\mathrm{Fun}(G, k)$ of k-valued functions on G, with comultiplication $\Delta(f)(x, y) = f(xy)$ (where we identify $H \otimes H$ with $\mathrm{Fun}(G \times G, k)$), and counit $\varepsilon(f) = f(1)$.

(ii) Show that $\mathrm{Fun}(G, k)^* = k[G]$, the monoid algebra of G (with basis $x \in G$ and product $x \cdot y = xy$), with coproduct $\Delta(x) = x \otimes x$, and counit $\varepsilon(x) = 1$, $x \in G$. Note that the bialgebra $k[G]$ may be defined for any G (not necessarily finite).

Exercise 1.21.6. Let H be a k-algebra, $\mathcal{C} = H-\mathbf{mod}$ be the category of H-modules, and $F : \mathcal{C} \to \mathbf{Vec}$ be the forgetful functor (we don't assume finite dimensionality). Assume that $\mathcal{C}$ is monoidal, and F is given a monoidal structure J. Show that this endows H with the structure of a bialgebra, such that (F, J) defines a monoidal equivalence $\mathcal{C} \to \mathbf{Rep}(H)$.

Note that not only modules, but also comodules over a bialgebra H form a monoidal category. Indeed, for a finite dimensional bialgebra, this is clear, as right (respectively, left) modules over H is the same thing as left (respectively, right) comodules over H^*. In general, if X, Y are, say, right H-comodules, then the right comodule $X \otimes Y$ is the usual tensor product of X, Y with the coaction map defined as follows: if $x \in X, y \in Y$, $\pi(x) = \sum x_i \otimes a_i$, $\pi(y) = \sum y_j \otimes b_j$, then

$$\pi_{X \otimes Y}(x \otimes y) = \sum x_i \otimes y_j \otimes a_i b_j.$$

For a bialgebra H, the monoidal category of right H-comodules will be denoted by $H-\mathbf{comod}$, and the subcategory of finite dimensional comodules by $H-\mathrm{comod}$.

1.22. **Hopf algebras.** Let us now consider the additional structure on the bialgebra $H = \mathsf{End}(F)$ from the previous subsection in the case when the category $\mathcal{C}$ has right duals. In this case, one can define a linear map $S : H \to H$ by the formula

$$S(a)_X = a^*_{X^*},$$

where we use the natural identification of $F(X)^*$ with $F(X^*)$.

Proposition 1.22.1. *("the antipode axiom") Let $\mu : H \otimes H \to H$ and $i : k \to H$ be the multiplication and the unit maps of H. Then*

$$\mu \circ (\mathrm{Id} \otimes S) \circ \Delta = i \circ \varepsilon = \mu \circ (S \otimes \mathrm{Id}) \circ \Delta$$

as maps $H \to H$.

Proof. For any $b \in \mathsf{End}(F \otimes F)$ the linear map $\mu \circ (\mathrm{Id} \otimes S)(\alpha^{-1}_{F,F}(b))_X$, $X \in \mathcal{C}$ is given by
(1.22.1)

$$F(X) \xrightarrow{\mathrm{coev}_{F(X)}} F(X) \otimes F(X)^* \otimes F(X) \xrightarrow{b_{X,X^*}} F(X) \otimes F(X)^* \otimes F(X) \xrightarrow{\mathrm{ev}_{F(X)}} F(X),$$

where we suppress the identity isomorphisms, the associativity constraint, and the isomorphism $F(X)^* \cong F(X^*)$. Indeed, it suffices to check (1.22.1) for $b = \eta \otimes \nu$, where $\eta, \nu \in H$, which is straightforward.

Now the first equality of the proposition follows from the commutativity of the diagram

(1.22.2)
$$\begin{tikzcd}
F(X) \arrow[r, "\mathrm{coev}_{F(X)}"] \arrow[d, "\mathrm{Id}"'] & F(X) \otimes F(X)^* \otimes F(X) \arrow[d, "J_{X,X^*}"'] \\
F(X) \arrow[r, "F(\mathrm{coev}_X)"] \arrow[d, "\eta_{\mathbf{1}}"'] & F(X \otimes X^*) \otimes F(X) \arrow[d, "\eta_{X \otimes X^*}"] \\
F(X) \arrow[r, "F(\mathrm{coev}_X)"] \arrow[d, "\mathrm{Id}"'] & F(X \otimes X^*) \otimes F(X) \arrow[d, "J^{-1}_{X,X^*}"] \\
F(X) & F(X) \otimes F(X)^* \otimes F(X), \arrow[l, "\mathrm{ev}_{F(X)}"']
\end{tikzcd}$$

for any $\eta \in \mathsf{End}(F)$.

Namely, the commutativity of the upper and the lower square follows from the fact that upon identification of $F(X)^*$ with $F(X^*)$, the morphisms $\mathrm{ev}_{F(X)}$ and $\mathrm{coev}_{F(X)}$ are given by the diagrams of Exercise 1.10.6. The middle square commutes by the naturality of η. The composition of left vertical arrows gives $\varepsilon(\eta)\mathrm{Id}_{F(X)}$, while the composition of the top, right, and bottom arrows gives $\mu \circ (\mathrm{Id} \otimes S) \circ \Delta(\eta)$.

The second equality is proved similarly. $\square$

Definition 1.22.2. An *antipode* on a bialgebra H is a linear map $S : H \to H$ which satisfies the equalities of Proposition 1.22.1.

Exercise 1.22.3. Show that the antipode axiom is self-dual in the following sense: if H is a finite dimensional bialgebra with antipode S_H, then the bialgebra H^* also admits an antipode $S_{H^*} = S_H^*$.

The following is a "linear algebra" analog of the fact that the right dual, when it exists, is unique up to a unique isomorphism.

Proposition 1.22.4. *An antipode on a bialgebra H is unique if exists.*

Proof. The proof essentially repeats the proof of uniqueness of right dual. Let S, S' be two antipodes for H. Then using the antipode properties of S, S', associativity of μ, and coassociativity of Δ, we get

$$S = \mu \circ (S \otimes [\mu \circ (\mathrm{Id} \otimes S') \circ \Delta]) \circ \Delta =$$

$$\mu \circ (\mathrm{Id} \otimes \mu) \circ (S \otimes \mathrm{Id} \otimes S') \circ (\mathrm{Id} \otimes \Delta) \circ \Delta =$$

$$\mu \circ (\mu \otimes \mathrm{Id}) \circ (S \otimes \mathrm{Id} \otimes S') \circ (\Delta \otimes \mathrm{Id}) \circ \Delta =$$

$$\mu \circ ([\mu \circ (S \otimes \mathrm{Id}) \circ \Delta] \otimes S') \circ \Delta = S'.$$

$\square$

Proposition 1.22.5. *If S is an antipode on a bialgebra H then S is an antihomomorphism of algebras with unit and of coalgebras with counit.*

Proof. Let

$$(\Delta \otimes \mathrm{Id}) \circ \Delta(a) = (\mathrm{Id} \otimes \Delta) \circ \Delta(a) = \sum_i a_i^1 \otimes a_i^2 \otimes a_i^3,$$

$$(\Delta \otimes \mathrm{Id}) \circ \Delta(b) = (\mathrm{Id} \otimes \Delta) \circ \Delta(b) = \sum_j b_j^1 \otimes b_j^2 \otimes b_j^3.$$

Then using the definition of the antipode, we have

$$S(ab) = \sum_i S(a_i^1 b) a_i^2 S(a_i^3) = \sum_{i,j} S(a_i^1 b_j^1) a_i^2 b_j^2 S(b_j^3) S(a_i^3) = S(b)S(a).$$

Thus S is an antihomomorphism of algebras (which is obviously unital). The fact that it is an antihomomorphism of coalgebras then follows using the self-duality of the axioms (see Exercises 1.21.4,1.22.3), or can be shown independently by a similar argument. $\square$

Corollary 1.22.6. *(i) If H is a bialgebra with an antipode S, then the abelian monoidal category $\mathcal{C} = \mathrm{Rep}(H)$ has right duals. Namely, for any object X, the right dual X^* is the usual dual space of X, with action of H given by*

$$\rho_{X^*}(a) = \rho_X(S(a))^*,$$

and the usual evaluation and coevaluation morphisms of the category Vec.

(ii) If in addition S is invertible, then $\mathcal{C}$ also admits left duals, i.e. is rigid (in other words, $\mathcal{C}$ is tensor category). Namely, for any object X, the left dual ${}^*X$ is the usual dual space of X, with action of H given by

$$\rho_{{}^*X}(a) = \rho_X(S^{-1}(a))^*,$$

and the usual evaluation and coevaluation morphisms of the category Vec.

Proof. Part (i) follows from the antipode axiom and Proposition 1.22.5. Part (ii) follows from part (i) and the fact that the operation of taking the left dual is inverse to the operation of taking the right dual. □

Remark 1.22.7. A similar statement holds for finite dimensional comodules. Namely, if X is a finite dimensional right comodule over a bialgebra H with an antipode, then the right dual is the usual dual X^* with

$$(\pi_{X^*}(f), x \otimes \phi) := ((\mathrm{Id} \otimes S)(\pi_X(x)), f \otimes \phi),$$

$x \in X, f \in X^*, \phi \in H^*$. If S is invertible, then the left dual ${}^*X$ is defined by the same formula with S replaced by S^{-1}.

Remark 1.22.8. The fact that S is an antihomomorphism of coalgebras is the "linear algebra" version of the categorical fact that dualization changes the order of tensor product (Proposition 1.10.7(ii)).

Definition 1.22.9. A bialgebra equipped with an invertible antipode S is called a *Hopf algebra*.

Remark 1.22.10. We note that many authors use the term "Hopf algebra" for any bialgebra with an antipode.

Thus, Corollary 1.22.6 states that if H is a Hopf algebra then $\mathrm{Rep}(H)$ is a tensor category. So, we get the following *reconstruction theorem for finite dimensional Hopf algebras.*

Theorem 1.22.11. *The assignments $(\mathcal{C}, F) \mapsto H = \mathsf{End}(F)$, $H \mapsto (\mathrm{Rep}(H), \mathrm{Forget})$ are mutually inverse bijections between*

1) finite tensor categories $\mathcal{C}$ with a fiber functor F, up to monoidal equivalence and isomorphism of monoidal functors;

2) finite dimensional Hopf algebras over k up to isomorphism.

Proof. Straightforward from the above. $\square$

Exercise 1.22.12. The algebra of functions $\mathrm{Fun}(G,k)$ on a finite monoid G is a Hopf algebra if and only if G is a group. In this case, the antipode is given by the formula $S(f)(x) = f(x^{-1})$, $x \in G$.

More generally, if G is an affine algebraic group over k, then the algebra $\mathcal{O}(G)$ of regular functions on G is a Hopf algebra, with the comultiplication, counit, and antipode defined as in the finite case.

Similarly, $k[G]$ is a Hopf algebra if and only if G is a group, with $S(x) = x^{-1}$, $x \in G$.

Exercises 1.21.5 and 1.22.12 motivate the following definition:

Definition 1.22.13. In any coalgebra C, a nonzero element $g \in C$ such that $\Delta(g) = g \otimes g$ is called a *grouplike element*.

Exercise 1.22.14. Show that if g is a grouplike of a Hopf algebra H, then g is invertible, with $g^{-1} = S(g)$. Also, show that the product of two grouplike elements is grouplike. In particular, grouplike elements of any Hopf algebra H form a group, denoted $\mathbf{G}(H)$. Show that this group can also be defined as the group of isomorphism classes of 1-dimensional H-comodules under tensor multiplication.

Proposition 1.22.15. *If H is a finite dimensional bialgebra with an antipode S, then S is invertible, so H is a Hopf algebra.*

Proof. Let H_n be the image of S^n. Since S is an antihomomorphism of algebras and coalgebras, H_n is a Hopf subalgebra of H. Let m be the smallest n such that $H_n = H_{n+1}$ (it exists because H is finite dimensional). We need to show that $m = 0$. If not, we can assume that $m = 1$ by replacing H with H_{m-1}.

We have a map $S' : H_1 \to H_1$ inverse to S. For $a \in H$, let the triple coproduct of a be

$$\sum_i a_i^1 \otimes a_i^2 \otimes a_i^3.$$

Consider the element

$$b = \sum_i S'(S(a_i^1))S(a_i^2)a_i^3.$$

On the one hand, collapsing the last two factors using the antipode axiom, we have $b = S'(S(a))$. On the other hand, writing b as

$$b = \sum_i S'(S(a_i^1))S(S'(S(a_i^2)))a_i^3$$

and collapsing the first two factors using the antipode axiom, we get $b = a$. Thus $a = S'(S(a))$ and thus $a \in H_1$, so $H = H_1$, a contradiction. $\square$

Exercise 1.22.16. Let μ^{op} and Δ^{op} be obtained from μ, Δ by permutation of components.

(i) Show that if $(H, \mu, i, \Delta, \varepsilon, S)$ is a Hopf algebra, then $H_{op} := (H, \mu^{op}, i, \Delta, \varepsilon, S^{-1})$, $H^{cop} := (H, \mu, i, \Delta^{op}, \varepsilon, S^{-1})$, $H^{cop}_{op} := (H, \mu^{op}, i, \Delta^{op}, \varepsilon, S)$ are Hopf algebras. Show that H is isomorphic to H^{cop}_{op}, and H_{op} to H^{cop}.

(ii) Suppose that a bialgebra H is a commutative ($\mu = \mu^{op}$) or cocommutative ($\Delta = \Delta^{op}$). Let S be an antipode on H. Show that $S^2 = 1$.

(iii) Assume that bialgebras H and H^{cop} have antipodes S and S'. Show that $S' = S^{-1}$, so H is a Hopf algebra.

Exercise 1.22.17. Show that if A, B are bialgebras, bialgebras with antipode, or Hopf algebras, then so is the tensor product $A \otimes B$.

Exercise 1.22.18. A finite dimensional module or comodule over a Hopf algebra is invertible if and only if it is 1-dimensional.

1.23. **Reconstruction theory in the infinite setting.** In this subsection we would like to generalize the reconstruction theory to the situation when the category $\mathcal{C}$ is not assumed to be finite.

Let $\mathcal{C}$ be any essentially small k-linear abelian category, and $F : \mathcal{C} \to \mathrm{Vec}$ an exact, faithful functor. In this case one can define the space $\mathrm{Coend}(F)$ as follows:

$$\mathrm{Coend}(F) := (\oplus_{X \in \mathcal{C}} F(X)^* \otimes F(X))/E$$

where E is spanned by elements of the form $y_* \otimes F(f)x - F(f)^* y_* \otimes x$, $x \in F(X)$, $y_* \in F(Y)^*$, $f \in \mathsf{Hom}(X, Y)$; in other words, $\mathrm{Coend}(F) = \varinjlim \mathsf{End}(F(X))^*$. Thus we have $\mathsf{End}(F) = \varprojlim \mathsf{End}(F(X)) = \mathrm{Coend}(F)^*$, which yields a coalgebra structure on $\mathrm{Coend}(F)$. So the algebra $\mathsf{End}(F)$ (which may be infinite dimensional) carries the inverse limit topology, in which a basis of neighborhoods of zero is formed by the kernels K_X of the maps $\mathsf{End}(F) \to \mathsf{End}(F(X))$, $X \in \mathcal{C}$, and $\mathrm{Coend}(F) = \mathsf{End}(F)^\vee$, the space of continuous linear functionals on $\mathsf{End}(F)$.

The following theorem is standard (see [Ta2]).

Theorem 1.23.1. *Let $\mathcal{C}$ be a k-linear abelian category with an exact faithful functor $F : \mathcal{C} \to Vec$. Then F defines an equivalence between $\mathcal{C}$ and the category of finite dimensional right comodules over* $C := \mathrm{Coend}(F)$ *(or, equivalently, with the category of continuous finite dimensional left* $\mathsf{End}(F)$*-modules).*

Proof. (sketch) Consider the ind-object $Q := \oplus_{X\in\mathcal{C}} F(X)^* \otimes X$. For $X, Y \in \mathcal{C}$ and $f \in \mathsf{Hom}(X,Y)$, let

$$j_f : F(Y)^* \otimes X \to F(X)^* \otimes X \oplus F(Y)^* \otimes Y \subset Q$$

be the morphism defined by the formula

$$j_f = \mathrm{Id} \otimes f - F(f)^* \otimes \mathrm{Id}.$$

Let I be the quotient of Q by the image of the direct sum of all j_f. In other words, $I = \varinjlim(F(X)^* \otimes X)$.

The following statements are easy to verify:

(i) I represents the functor $F(\bullet)^*$, i.e. $\mathsf{Hom}(X, I)$ is naturally isomorphic to $F(X)^*$; in particular, I is injective.

(ii) $F(I) = C$, and I is naturally a left C-comodule (the comodule structure is induced by the coevaluation morphism $F(X)^* \otimes X \to F(X)^* \otimes F(X) \otimes F(X)^* \otimes X$).

(iii) Let us regard F as a functor $\mathcal{C} \to C - \text{comod}$. For $M \in C - \text{comod}$, let $\theta_M : M \otimes I \to M \otimes C \otimes I$ be the morphism $\pi_M \otimes \mathrm{Id} - \mathrm{Id} \otimes \pi_I$, and let K_M be the kernel of θ_M. Then the functor $G : C - \text{comod} \to \mathcal{C}$ given by the formula $G(M) = \mathrm{Ker}\theta_M$, is a quasi-inverse to F.

This completes the proof. □

Now assume that the abelian category $\mathcal{C}$ is also monoidal. Then the coalgebra $\mathrm{Coend}(F)$ also carries a multiplication and unit, dual to the comultiplication and counit of $\mathsf{End}(F)$. More precisely, since $\mathsf{End}(F)$ may now be infinite dimensional, the algebra $\mathsf{End}(F \otimes F)$ is in general isomorphic not to the usual tensor product $\mathsf{End}(F)\otimes\mathsf{End}(F)$, but rather to its completion $\mathsf{End}(F)\widehat{\otimes}\,\mathsf{End}(F)$ with respect to the inverse limit topology. Thus the comultiplication of $\mathsf{End}(F)$ is a continuous linear map $\Delta : \mathsf{End}(F) \to \mathsf{End}(F)\widehat{\otimes}\,\mathsf{End}(F)$. The dual Δ^* of this map defines a multiplication on $\mathrm{Coend}(F)$.

If $\mathcal{C}$ has right duals, the bialgebra $\mathrm{Coend}(F)$ acquires an antipode, defined in the same way as in the finite dimensional case. This antipode is invertible if there are also left duals (i.e. if $\mathcal{C}$ is rigid). Thus Theorem 1.23.1 implies the following “infinite” extensions of the reconstruction theorems.

Theorem 1.23.2. *The assignments* $(\mathcal{C}, F) \mapsto H = \mathrm{Coend}(F)$*,* $H \mapsto (H - \mathrm{Comod}, \mathrm{Forget})$ *are mutually inverse bijections between*

1) k-linear abelian monoidal categories $\mathcal{C}$ with a fiber functor F, up to monoidal equivalence and isomorphism of monoidal functors, and bialgebras over k, up to isomorphism;

2) k-linear abelian monoidal categories $\mathcal{C}$ with right duals with a fiber functor F, up to monoidal equivalence and isomorphism of monoidal functors, and bialgebras over k with an antipode, up to isomorphism;

3) tensor categories $\mathcal{C}$ over k with a fiber functor F, up to monoidal equivalence and isomorphism of monoidal functors, and Hopf algebras over k, up to isomorphism.

Remark 1.23.3. This theorem allows one to give a categorical proof of Proposition 1.22.4, deducing it from the fact that the right dual, when it exists, is unique up to a unique isomorphism.

Remark 1.23.4. Corollary 1.22.15 is not true, in general, in the infinite dimensional case: there exist bialgebras H with a non-invertible antipode S, see [Ta1]. Therefore, there exist ring categories with simple object $\mathbf{1}$ and right duals that do not have left duals, i.e., are not tensor categories (namely, $H-\mathrm{comod}$).

In the next few subsections, we will review some of the most important basic results about Hopf algebras. For a much more detailed treatment, see the book [Mo].

1.24. **More examples of Hopf algebras.** Let us give a few more examples of Hopf algebras. As we have seen, to define a Hopf algebra, it suffices to give an associative unital algebra H, and define a coproduct on generators of H (this determines a Hopf algebra structure on H uniquely if it exists). This is what we'll do in the examples below.

Example 1.24.1. (Enveloping algebras) Let $\mathfrak{g}$ be a Lie algebra, and let $H = U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$. Define the coproduct on H by setting $\Delta(x) = x\otimes 1+1\otimes x$ for all $x \in \mathfrak{g}$. It is easy to show that this extends to the whole H, and that H equipped with this Δ is a Hopf algebra. Moreover, it is easy to see that the tensor category $\mathrm{Rep}(H)$ is equivalent to the tensor category $\mathrm{Rep}(\mathfrak{g})$.

This example motivates the following definition.

Definition 1.24.2. An element x of a bialgebra H is called *primitive* if $\Delta(x) = x\otimes 1+1\otimes x$. The space of primitive elements of H is denoted $\mathrm{Prim}(H)$.

Exercise 1.24.3. (i) Show that $\mathrm{Prim}(H)$ is a Lie algebra under the commutator.

(ii) Show that if x is a primitive element then $\varepsilon(x) = 0$, and in presence of an antipode $S(x) = -x$.

Exercise 1.24.4. (i) Let V be a vector space, and SV be the symmetric algebra V. Then SV is a Hopf algebra (namely, it is the universal enveloping algebra of the abelian Lie algebra V). Show that if k has characteristic zero, then $\mathrm{Prim}(SV) = V$.

(ii) What happens in characteristic p?

Hint. One can restrict to a situation when V is finite dimensional. In this case, regarding elements $f \in SV$ as polynomials on V^*, one can show that f is primitive if and only if it is additive, i.e., $f(x+y) = f(x) + f(y)$.

(iii) Let $\mathfrak{g}$ be a Lie algebra over a field of characteristic zero. Show that $\mathrm{Prim}(U(\mathfrak{g})) = \mathfrak{g}$.

Hint. Identify $U(\mathfrak{g})$ with $S\mathfrak{g}$ as coalgebras by using the symmetrization map.

Example 1.24.5. (Taft algebras) Let q be a primitive n-th root of unity. Let H be the algebra (of dimension n^2) generated over k by g and x satisfying the following relations: $g^n = 1$, $x^n = 0$ and $gxg^{-1} = qx$. Define the coproduct on H by $\Delta(g) = g \otimes g$, $\Delta(x) = x \otimes g + 1 \otimes x$. It is easy to show that this extends to a Hopf algebra structure on H. This Hopf algebra H is called the *Taft algebra*. For $n = 2$, one obtains the Sweedler Hopf algebra of dimension 4. Note that H is not commutative or cocommutative, and $S^2 \neq 1$ on H (as $S^2(x) = qx$).

This example motivates the following generalization of Definition 1.24.2.

Definition 1.24.6. Let g, h be grouplike elements of a coalgebra H. A skew-primitive element of type (h, g) is an element $x \in H$ such that $\Delta(x) = h \otimes x + x \otimes g$.

Remark 1.24.7. A multiple of $h - g$ is always a skew-primitive element of type (h, g). Such a skew-primitive element is called *trivial*. Note that the element x in Example 1.24.5 is nontrivial.

Exercise 1.24.8. Let x be a skew-primitive element of type h, g in a Hopf algebra H.

(i) Show that $\varepsilon(x) = 0$, $S(x) = -h^{-1}xg^{-1}$.

(ii) Show that if $a, b \in H$ are grouplike elements, then axb is a skew-primitive element of type (ahb, agb).

Example 1.24.9. (Nichols Hopf algebras) Let $H = \mathbb{C}[\mathbb{Z}/2\mathbb{Z}] \ltimes \wedge(x_1, ..., x_n)$, where the generator g of $\mathbb{Z}/2\mathbb{Z}$ acts on x_i by $gx_ig^{-1} = -x_i$. Define the coproduct on H by making g grouplike, and setting $\Delta(x_i) := x_i \otimes g + 1 \otimes x_i$ (so x_i are skew-primitive elements). Then H is a Hopf algebra of dimension 2^{n+1}. For $n = 1$, H is the Sweedler Hopf algebra from the previous example.

Exercise 1.24.10. Show that the Hopf algebras of Examples 1.24.1,1.24.5,1.24.9 are well defined.

Exercise 1.24.11. (Semidirect product Hopf algebras) Let H be a Hopf algebra, and G a group of automorphisms of H. Let A be the semidirect product $k[G] \ltimes H$. Show that A admits a unique structure of a Hopf algebra in which $k[G]$ and H are Hopf subalgebras.