1.4. **Monoidal functors, equivalence of monoidal categories.** As we have explained, monoidal categories are a categorification of monoids. Now we pass to categorification of morphisms between monoids, namely *monoidal functors*.

Definition 1.4.1. Let $(\mathcal{C}, \otimes, \mathbf{1}, a, \iota)$ and $(\mathcal{C}', \otimes', \mathbf{1}', a', \iota')$ be two monoidal categories. A *monoidal functor* from $\mathcal{C}$ to $\mathcal{C}'$ is a pair (F, J) where $F : \mathcal{C} \to \mathcal{C}'$ is a functor, and $J = \{J_{X,Y} : F(X) \otimes' F(Y) \xrightarrow{\sim} F(X \otimes Y) | X, Y \in \mathcal{C}\}$ is a natural isomorphism, such that $F(\mathbf{1})$ is isomorphic to $\mathbf{1}'$. and the diagram
(1.4.1)

$$\begin{tikzcd}
(F(X) \otimes' F(Y)) \otimes' F(Z) \arrow[rr, "a'_{F(X),F(Y),F(Z)}"] \arrow[d, "J_{X,Y} \otimes' \mathrm{Id}_{F(Z)}"'] & & F(X) \otimes' (F(Y) \otimes' F(Z)) \arrow[d, "\mathrm{Id}_{F(X)} \otimes' J_{Y,Z}"'] \\
F(X \otimes Y) \otimes' F(Z) \arrow[d, "J_{X \otimes Y, Z}"'] & & F(X) \otimes' F(Y \otimes Z) \arrow[d, "J_{X, Y \otimes Z}"'] \\
F((X \otimes Y) \otimes Z) \arrow[rr, "F(a_{X,Y,Z})"] & & F(X \otimes (Y \otimes Z))
\end{tikzcd}$$

is commutative for all $X, Y, Z \in \mathcal{C}$ ("the monoidal structure axiom").

A monoidal functor F is said to be an *equivalence of monoidal categories* if it is an equivalence of ordinary categories.

Remark 1.4.2. It is important to stress that, as seen from this definition, a monoidal functor is not just a functor between monoidal categories, but a functor with an additional structure (the isomorphism J) satisfying a certain equation (the monoidal structure axiom). As we will see later, this equation may have more than one solution, so the same functor can be equipped with different monoidal structures.

It turns out that if F is a monoidal functor, then there is a canonical isomorphism $\varphi : \mathbf{1}' \to F(\mathbf{1})$. This isomorphism is defined by the commutative diagram

$$\begin{tikzcd}
\mathbf{1}' \otimes' F(\mathbf{1}) \arrow[r, "l'_{F(\mathbf{1})}"] \arrow[d, "\varphi \otimes' \mathrm{Id}_{F(X)}"'] & F(\mathbf{1}) \\
F(\mathbf{1}) \otimes' F(\mathbf{1}) \arrow[r, "J_{\mathbf{1},\mathbf{1}}"] & F(\mathbf{1} \otimes \mathbf{1}) \arrow[u, "F(l_{\mathbf{1}})"']
\end{tikzcd} \tag{1.4.2}$$

where l, r, l', r' are the unit isomorphisms for $\mathcal{C}, \mathcal{C}'$ defined in Subsection 1.2.

Proposition 1.4.3. *For any monoidal functor* $(F, J) : \mathcal{C} \to \mathcal{C}'$, *the diagrams*

$$\begin{tikzcd} \mathbf{1}' \otimes' F(X) \arrow[r, "l'_{F(X)}"] \arrow[d, "\varphi \otimes' \mathrm{Id}_{F(X)}"'] & F(X) \\ F(\mathbf{1}) \otimes' F(X) \arrow[r, "J_{\mathbf{1},X}"] & F(\mathbf{1} \otimes X) \arrow[u, "F(l_X)"'] \end{tikzcd} \tag{1.4.3}$$

and

$$\begin{tikzcd} F(X) \otimes' \mathbf{1}' \arrow[r, "r'_{F(X)}"] \arrow[d, "\mathrm{Id}_{F(X)} \otimes' \varphi"'] & F(X) \\ F(X) \otimes' F(\mathbf{1}) \arrow[r, "J_{X,\mathbf{1}}"] & F(X \otimes \mathbf{1}) \arrow[u, "F(r_X)"'] \end{tikzcd} \tag{1.4.4}$$

are commutative for all $X \in \mathcal{C}$.

Exercise 1.4.4. Prove Proposition 1.4.3.

Proposition 1.4.3 implies that a monoidal functor can be equivalently defined as follows.

Definition 1.4.5. A monoidal functor $\mathcal{C} \to \mathcal{C}'$ is a triple (F, J, φ) which satisfies the monoidal structure axiom and Proposition 1.4.3.

This is a more traditional definition of a monoidal functor.

Remark 1.4.6. It can be seen from the above that for any monoidal functor (F, J) one can safely identify $\mathbf{1}'$ with $F(\mathbf{1})$ using the isomorphism φ, and assume that $F(\mathbf{1}) = \mathbf{1}'$ and $\varphi = \mathrm{Id}$ (similarly to how we have identified $\mathbf{1} \otimes X$ and $X \otimes \mathbf{1}$ with X and assumed that $l_X = r_X = \mathrm{Id}_X$). We will usually do so from now on. Proposition 1.4.3 implies that with these conventions, one has

$$J_{\mathbf{1},X} = J_{X,\mathbf{1}} = \mathrm{Id}_X. \tag{1.4.5}$$

Remark 1.4.7. It is clear that the composition of monoidal functors is a monoidal functor. Also, the identity functor has a natural structure of a monoidal functor.

1.5. **Morphisms of monoidal functors.** Monoidal functors between two monoidal categories themselves form a category. Namely, one has the following notion of a morphism (or natural transformation) between two monoidal functors.

Definition 1.5.1. Let $(\mathcal{C}, \otimes, \mathbf{1}, a, \iota)$ and $(\mathcal{C}', \otimes', \mathbf{1}', a', \iota')$ be two monoidal categories, and (F^1, J^1), (F^2, J^2) two monoidal functors from $\mathcal{C}$ to $\mathcal{C}'$. A *morphism* (or a *natural transformation*) of monoidal functors $\eta : (F^1, J^1) \to (F^2, J^2)$ is a natural transformation $\eta : F^1 \to F^2$ such that $\eta_{\mathbf{1}}$ is an isomorphism, and the diagram

$$\begin{array}{ccc} F^1(X) \otimes' F^1(Y) & \xrightarrow{J^1_{X,Y}} & F^1(X \otimes Y) \\ {\scriptstyle \eta_X \otimes' \eta_Y} \downarrow & & \downarrow {\scriptstyle \eta_{X \otimes Y}} \\ F^2(X) \otimes' F^2(Y) & \xrightarrow{J^2_{X,Y}} & F^2(X \otimes Y) \end{array} \tag{1.5.1}$$

is commutative for all $X, Y \in \mathcal{C}$.

Remark 1.5.2. It is easy to show that $\eta_{\mathbf{1}} \circ \varphi^1 = \varphi^2$, so if one makes the convention that $\varphi^i = \mathrm{Id}$, one has $\eta_{\mathbf{1}} = \mathrm{Id}$.

Remark 1.5.3. It is easy to show that if $F : \mathcal{C} \to \mathcal{C}'$ is an equivalence of monoidal categories, then there exists a monoidal equivalence $F^{-1} : \mathcal{C}' \to \mathcal{C}$ such that the functors $F \circ F^{-1}$ and $F^{-1} \circ F$ are isomorphic to the identity functor as monoidal functors. Thus, for any monoidal category $\mathcal{C}$, the monoidal auto-equivalences of $\mathcal{C}$ up to isomorphism form a group with respect to composition.

1.6. **Examples of monoidal functors.** Let us now give some examples of monoidal functors and natural transformations.

Example 1.6.1. An important class of examples of monoidal functors is *forgetful functors* (e.g. functors of "forgetting the structure", from the categories of groups, topological spaces, etc., to the category of sets). Such functors have an obvious monoidal structure. An example important in these notes is the forgetful functor $\mathbf{Rep}_G \to \mathbf{Vec}$ from the representation category of a group to the category of vector spaces. More generally, if $H \subset G$ is a subgroup, then we have a forgetful (or restriction) functor $\mathbf{Rep}_G \to \mathbf{Rep}_H$. Still more generally, if $f : H \to G$ is a group homomorphism, then we have the pullback functor $f^* : \mathbf{Rep}_G \to \mathbf{Rep}_H$. All these functors are monoidal.

Example 1.6.2. Let $f : H \to G$ be a homomorphism of groups. Then any H-graded vector space is naturally G-graded (by pushforward of grading). Thus we have a natural monoidal functor $f_* : \mathbf{Vec}_H \to \mathbf{Vec}_G$. If G is the trivial group, then f_* is just the forgetful functor $\mathbf{Vec}_H \to \mathbf{Vec}$.

Example 1.6.3. Let A be a k-algebra with unit, and $\mathcal{C} = A - \mathbf{mod}$ be the category of left A-modules. Then we have a functor $F : A -$

bimod $\to \mathrm{End}(\mathcal{C})$ given by $F(M) = M\otimes_A$. This functor is naturally monoidal. A similar functor $F : A - \mathrm{bimod} \to \mathsf{End}(\mathcal{C})$. can be defined if A is a finite dimensional k-algebra, and $\mathcal{C} = A - \mathrm{mod}$ is the category of finite dimensional left A-modules.

Proposition 1.6.4. *The functor* $F : A-\mathrm{bimod} \to \mathrm{End}(\mathcal{C})$ *takes values in the full monoidal subcategory* $\mathrm{End}_{re}(\mathcal{C})$ *of right exact endofunctors of* $\mathcal{C}$*, and defines an equivalence between monoidal categories* $A - \mathrm{bimod}$ *and* $\mathrm{End}_{re}(\mathcal{C})$

Proof. The first statement is clear, since the tensor product functor is right exact. To prove the second statement, let us construct the quasi-inverse functor F^{-1}. Let $G \in \mathrm{End}_{re}(\mathcal{C})$. Define $F^{-1}(G)$ by the formula $F^{-1}(G) = G(A)$; this is clearly an A-bimodule, since it is a left A-module with a commuting action $\mathrm{End}_A(A) = A^{op}$ (the opposite algebra). We leave it to the reader to check that the functor F^{-1} is indeed quasi-inverse to F. $\square$

Remark 1.6.5. A similar statement is valid without the finite dimensionality assumption, if one adds the condition that the right exact functors must commute with inductive limits.

Example 1.6.6. Let S be a monoid, and $\mathcal{C} = \mathrm{Vec}_S$, and $\mathsf{Id}_{\mathcal{C}}$ the identity functor of $\mathcal{C}$. It is easy to see that morphisms $\eta : \mathsf{Id}_{\mathcal{C}} \to \mathsf{Id}_{\mathcal{C}}$ correspond to homomorphisms of monoids: $\eta : S \to k$ (where k is equipped with the multiplication operation). In particular, $\eta(s)$ may be 0 for some s, so η does not have to be an isomorphism.

1.7. **Monoidal functors between categories $\mathcal{C}_G^\omega$.** Let G_1, G_2 be groups, A an abelian group, and $\omega_i \in Z^3(G_i, A)$, $i = 1, 2$ be 3-cocycles. Let $\mathcal{C}_i = \mathcal{C}_{G_i}^{\omega_i}$, $i = 1, 2$ (see Example 1.3.7).

Any monoidal functor $F : \mathcal{C}_1 \to \mathcal{C}_2$ defines, by restriction to simple objects, a group homomorphism $f : G_1 \to G_2$. Using the axiom (1.4.1) of a monoidal functor we see that a monoidal structure on F is given by

$$J_{g,h} = \mu(g, h)\mathrm{Id}_{\delta_{f(gh)}} : F(\delta_g) \otimes F(\delta_h) \xrightarrow{\sim} F(\delta_{gh}),\ g, h \in G_1, \tag{1.7.1}$$

where $\mu : G_1 \times G_1 \to A$ is a function such that

$$\omega_1(g, h, l)\mu(gh, l)\mu(g, h) = \mu(g, hl)\mu(h, l)\omega_2(f(g), f(h), f(l)),$$

for all $g, h, l \in G_1$. That is,

$$f^*\omega_2 = \omega_1\partial_2(\mu), \tag{1.7.2}$$

i.e., ω_1 and $f^*\omega_2$ are cohomologous in $Z^3(G_1, A)$.

Conversely, given a group homomorphism $f : G_1 \to G_2$, a function $\mu : G_1 \times G_1 \to A$ satisfying (1.7.2) gives rise to a monoidal functor $F : \mathcal{C}_1 \to \mathcal{C}_2$ defined by $F(\delta_g) = \delta_{f(g)}$ with the monoidal structure given by formula (1.7.1). This functor is an equivalence if and only if f is an isomorphism.

To summarize, monoidal functors $\mathcal{C}_{G_1}^{\omega_1} \to \mathcal{C}_{G_2}^{\omega_2}$ correspond to pairs (f, μ), where $f : G_1 \to G_2$ is a group homomorphism such that ω_1 and $f^*\omega_2$ are cohomologous, and μ is a function satisfying (1.7.2) (such functions are in a (non-canonical) bijection with A-valued 2-cocycles on G_1). Let $F_{f,\mu}$ denote the corresponding functor.

Let us determine natural monoidal transformations between $F_{f,\mu}$ and $F_{f',\mu'}$. Clearly, such a transformation exists if and only if $f = f'$, is always an isomorphism, and is determined by a collection of morphisms $\eta_g : \delta_{f(g)} \to \delta_{f(g)}$ (i.e., $\eta_g \in A$), satisfying the equation

$$\mu'(g,h)(\eta_g \otimes \eta_h) = \eta_{gh}\mu(g,h) \tag{1.7.3}$$

for all $g, h \in G_1$, i.e.,

$$\mu' = \mu\partial_1(\eta). \tag{1.7.4}$$

Conversely, every function $\eta : G_1 \to A$ satisfying (1.7.4) gives rise to a morphism of monoidal functors $\eta : F_{f,\mu} \to F_{f,\mu'}$ defined as above. Thus, functors $F_{f,\mu}$ and $F_{f',\mu'}$ are isomorphic as monoidal functors if and only if $f = f'$ and μ is cohomologous to μ'.

Thus, we have obtained the following proposition.

Proposition 1.7.1. *(i) The monoidal isomorphisms $F_{f,\mu} \to F_{f,\mu'}$ of monoidal functors $F_{f,\mu_i} : \mathcal{C}_{G_1}^{\omega_1} \to \mathcal{C}_{G_2}^{\omega_2}$ form a torsor over the group $H^1(G_1, k^\times) = \mathrm{Hom}(G_1, k^\times)$ of characters of G_1;*

(ii) Given f, the set of μ parametrizing isomorphism classes of $F_{f,\mu}$ is a torsor over $H^2(G_1, k^\times)$;

(iii) The structures of a monoidal category on $(\mathcal{C}_G, \otimes)$ are parametrized by $H^3(G, k^\times)/\mathrm{Out}(G)$, where $\mathrm{Out}(G)$ is the group of outer automorphisms of G. [5]

Remark 1.7.2. The same results, including Proposition 1.7.1, are valid if we replace the categories $\mathcal{C}_G^\omega$ by their "linear spans" Vec_G^ω, and require that the monoidal functors we consider are additive. To see this, it is enough to note that by definition, for any morphism η of monoidal functors, $\eta_1 \neq 0$, so equation (1.7.3) (with $h = g^{-1}$) implies

[5] Recall that the group $\mathrm{Inn}(G)$ of inner automorphisms of a group G acts trivially on $H^*(G, A)$ (for any coefficient group A), and thus the action of the group $\mathrm{Aut}(G)$ on $H^*(G, A)$ factors through $\mathrm{Out}(G)$.

that all η_g must be nonzero. Thus, if a morphism $\eta : F_{f,\mu} \to F_{f',\mu'}$ exists, then it is an isomorphism, and we must have $f = f'$.

Remark 1.7.3. The above discussion implies that in the definition of the categories $\mathcal{C}_G^\omega$ and Vec_G^ω, it may be assumed without loss of generality that the cocycle ω is normalized, i.e., $\omega(g, 1, h) = 1$, and thus $l_{\delta_g} = r_{\delta_g} = \mathrm{Id}$ (which is convenient in computations). Indeed, we claim that any 3-cocycle ω is cohomologous to a normalized one. To see this, it is enough to alter ω by dividing it by $\partial_2\mu$, where μ is any 2-cochain such that $\mu(g, 1) = \omega(g, 1, 1)$, and $\mu(1, h) = \omega(1, 1, h)^{-1}$.

Example 1.7.4. Let $G = \mathbb{Z}/n\mathbb{Z}$ where $n > 1$ is an integer, and $k = \mathbb{C}$. Consider the cohomology of $\mathbb{Z}/n\mathbb{Z}$.

Since $H^i(\mathbb{Z}/n\mathbb{Z}, \mathbb{C}) = 0$ for all $i > 0$, writing the long exact sequence of cohomology for the short exact sequence of coefficient groups

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{C} \longrightarrow \mathbb{C}^\times = \mathbb{C}/\mathbb{Z} \longrightarrow 0,$$

we obtain a natural isomorphism $H^i(\mathbb{Z}/n\mathbb{Z}, \mathbb{C}^\times) \cong H^{i+1}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z})$.

It is well known [Br] that the graded ring $H^*(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z})$ is $(\mathbb{Z}/n\mathbb{Z})[x]$ where x is a generator in degree 2. Moreover, as a module over $\mathrm{Aut}(\mathbb{Z}/n\mathbb{Z}) = (\mathbb{Z}/n\mathbb{Z})^\times$, we have $H^2(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) \cong H^1(\mathbb{Z}/n\mathbb{Z}, \mathbb{C}^\times) = (\mathbb{Z}/n\mathbb{Z})^\vee$. Therefore, using the graded ring structure, we find that $H^{2m}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) \cong H^{2m-1}(\mathbb{Z}/n\mathbb{Z}, \mathbb{C}^\times) = ((\mathbb{Z}/n\mathbb{Z})^\vee)^{\otimes m}$ as an $\mathrm{Aut}(\mathbb{Z}/n\mathbb{Z})$-module. In particular, $H^3(\mathbb{Z}/n\mathbb{Z}, \mathbb{C}^\times) = ((\mathbb{Z}/n\mathbb{Z})^\vee)^{\otimes 2}$.

This consideration shows that if $n = 2$ then the categorification problem has 2 solutions (the cases of trivial and non-trivial cocycle), while if n is a prime greater than 2 then there are 3 solutions: the trivial cocycle, and two non-trivial cocycles corresponding (non-canonically) to quadratic residues and non-residues $\bmod n$.

Let us give an explicit formula for the 3-cocycles on $\mathbb{Z}/n\mathbb{Z}$. Modulo coboundaries, these cocycles are given by

$$\phi(i, j, k) = \varepsilon^{\frac{si(j+k-(j+k)')}{n}}, \tag{1.7.5}$$

where ε is a primitive nth root of unity, $s \in \mathbb{Z}/n\mathbb{Z}$, and for an integer m we denote by m' the remainder of division of m by n.

Exercise 1.7.5. Show that when s runs over $\mathbb{Z}/n\mathbb{Z}$ this formula defines cocycles representing all cohomology classes in $H^3(\mathbb{Z}/n\mathbb{Z}, \mathbb{C}^\times)$.

1.8. **MacLane's strictness theorem.** As we have seen above, it is much simpler to work with monoidal categories in which the associativity and unit constrains are the identity maps.

Definition 1.8.1. A monoidal category $\mathcal{C}$ is *strict* if for all objects X, Y, Z in $\mathcal{C}$ one has equalities $(X \otimes Y) \otimes Z = X \otimes (Y \otimes Z)$ and

$X \otimes \mathbf{1} = X = \mathbf{1} \otimes X$, and the associativity and unit constraints are the identity maps.

Example 1.8.2. The category $\mathrm{End}(\mathcal{C})$ endofunctors of a category $\mathcal{C}$ is strict.

Example 1.8.3. Let $\overline{\text{Sets}}$ be the category whose objects are nonnegative integers, and $\mathsf{Hom}(m, n)$ is the set of maps from $\{0, ..., m-1\}$ to $\{0, ..., n-1\}$. Define the tensor product functor on objects by $m \otimes n = mn$, and for $f_1 : m_1 \to n_1, f_2 : m_2 \to n_2$, define $f_1 \otimes f_2 : m_1 m_2 \to n_1 n_2$ by

$$(f_1 \otimes f_2)(m_2 x + y) = n_2 f_1(x) + f_2(y), 0 \leq x \leq m_1 - 1,\ 0 \leq y \leq m_2 - 1.$$

Then $\overline{\text{Sets}}$ is a strict monoidal category. Moreover, we have a natural inclusion $\overline{\text{Sets}} \hookrightarrow \text{Sets}$, which is obviously a monoidal equivalence.

Example 1.8.4. This is really a linear version of the previous example. Let $k - \overline{\text{Vec}}$ be the category whose objects are nonnegative integers, and $\mathsf{Hom}(m, n)$ is the set of matrices with m columns and n rows over some field k (and the composition of morphisms is the product of matrices). Define the tensor product functor on objects by $m \otimes n = mn$, and for $f_1 : m_1 \to n_1, f_2 : m_2 \to n_2$, define $f_1 \otimes f_2 : m_1 m_2 \to n_1 n_2$ to be the Kronecker product of f_1 and f_2. Then $k - \overline{\text{Vec}}$ is a strict monoidal category. Moreover, we have a natural inclusion $k - \overline{\text{Vec}} \hookrightarrow k - \text{Vec}$, which is obviously a monoidal equivalence.

Similarly, for any group G one can define a strict monoidal category $k - \overline{\text{Vec}}_G$, whose objects are $\mathbb{Z}_+$-valued functions on G with finitely many nonzero values, and which is monoidally equivalent to $k - \text{Vec}_G$. We leave this definition to the reader.

On the other hand, some of the most important monoidal categories, such as **Sets**, **Vec**, $\mathbf{Vec}_G$, Sets, Vec, Vec_G, should be regarded as nonstrict (at least if one defines them in the usual way). It is even more indisputable that the categories $\mathbf{Vec}_G^\omega$, Vec_G^ω for cohomologically nontrivial ω are not strict.

However, the following remarkable theorem of MacLane implies that in practice, one may always assume that a monoidal category is strict.

Theorem 1.8.5. *Any monoidal category is monoidally equivalent to a strict monoidal category.*

Proof. The proof presented below was given in [JS]. We will establish an equivalence between $\mathcal{C}$ and the monoidal category of right $\mathcal{C}$-module endofunctors of $\mathcal{C}$, which we will discuss in more detail later. The noncategorical algebraic counterpart of this result is of course the fact that every monoid M is isomorphic to the monoid consisting of maps from M to itself commuting with the right multiplication.

For a monoidal category $\mathcal{C}$, let $\mathcal{C}'$ be the monoidal category defined as follows. The objects of $\mathcal{C}$ are pairs (F, c) where $F : \mathcal{C} \to \mathcal{C}$ is a functor and

$$c_{X,Y} : F(X) \otimes Y \xrightarrow{\sim} F(X \otimes Y)$$

is a functorial isomorphism, such that the following diagram is commutative for all objects X, Y, Z in $\mathcal{C}$:
(1.8.1)

$$\begin{tikzcd}
 & (F(X)\otimes Y)\otimes Z \arrow[dl, "c_{X,Y}\otimes \mathrm{Id}_Z"'] \arrow[dr, "a_{F(X),Y,Z}"] & \\
F(X\otimes Y)\otimes Z \arrow[d, "c_{X\otimes Y,Z}"'] & & F(X)\otimes(Y\otimes Z) \arrow[d, "c_{X,Y\otimes Z}"] \\
F((X\otimes Y)\otimes Z) \arrow[rr, "F(a_{X,Y,Z})"'] & & F(X\otimes(Y\otimes Z)).
\end{tikzcd}$$

A morphism $\theta : (F^1, c^1) \to (F^2, c^2)$ in $\mathcal{C}'$ is a natural transformation $\theta : F^1 \to F^2$ such that the following square commutes for all objects X, Y in $\mathcal{C}$:

(1.8.2)
$$\begin{tikzcd}
F^1(X)\otimes Y \arrow[r, "c^1_{X,Y}"] \arrow[d, "\theta_X\otimes \mathrm{Id}_Y"'] & F^1(X\otimes Y) \arrow[d, "\theta_{X\otimes Y}"] \\
F_2(X)\otimes Y \arrow[r, "c^2_{X,Y}"'] & F^2(X\otimes Y)
\end{tikzcd}$$

Composition of morphisms is the vertical composition of natural transformations. The tensor product of objects is given by $(F^1, c^1) \otimes (F^2, c^2) = (F^1F^2, c)$ where c is given by a composition

$$(1.8.3)\quad F^1F^2(X)\otimes Y \xrightarrow{c^1_{F_2(X),Y}} F^1(F^2(X)\otimes Y) \xrightarrow{F_1(c^2_{X,Y})} F^1F^2(X\otimes Y)$$

for all $X, Y \in \mathcal{C}$, and the tensor product of morphisms is the horizontal composition of natural transformations. Thus $\mathcal{C}'$ is a strict monoidal category (the unit object is the identity functor).

Consider now the functor of left multiplication $L : \mathcal{C} \to \mathcal{C}'$ given by

$$L(X) = (X \otimes \bullet,\ a_{X,\bullet,\bullet}),\ L(f) = f \otimes \bullet.$$

Note that the diagram (1.8.1) for $L(X)$ is nothing but the pentagon diagram (1.1.2).

We claim that this functor L is a monoidal equivalence.

First of all, L essentially surjective: it is easy to check that for any $(F, c) \in \mathcal{C}'$, (F, c) is isomorphic to $L(F(\mathbf{1}))$.

Let us now show that L is fully faithful. Let $\theta : L(X) \to L(Y)$ be a morphism in $\mathcal{C}$. Define $f : X \to Y$ to be the composite

$$X \xrightarrow{r_X^{-1}} X \otimes \mathbf{1} \xrightarrow{\theta_{\mathbf{1}}} Y \otimes \mathbf{1} \xrightarrow{r_Y} Y. \tag{1.8.4}$$

We claim that for all Z in $\mathcal{C}$ one has $\theta_Z = f \otimes \mathrm{Id}_Z$ (so that $\theta = L(f)$ and L is full). Indeed, this follows from the commutativity of the diagram (1.8.5)

$$\begin{tikzcd}
X \otimes Z \arrow[r, "r_X^{-1}\otimes \mathrm{Id}_Z"] \arrow[d, "f\otimes \mathrm{Id}_Z"'] & (X \otimes \mathbf{1}) \otimes Z \arrow[r, "a_{X,\mathbf{1},Z}"] \arrow[d, "\theta_{\mathbf{1}}\otimes Z"'] & X \otimes (\mathbf{1} \otimes Z) \arrow[r, "\mathrm{Id}_X\otimes l_Z"] \arrow[d, "\theta_{\mathbf{1}\otimes \mathrm{Id}_Z}"'] & X \otimes Z \arrow[d, "\theta_Z"'] \\
Y \otimes Z \arrow[r, "r_Y^{-1}\otimes \mathrm{Id}_Z"'] & (Y \otimes \mathbf{1}) \otimes Z \arrow[r, "a_{Y,\mathbf{1},Z}"'] & Y \otimes (\mathbf{1} \otimes Z) \arrow[r, "\mathrm{Id}_Y\otimes l_Z"'] & Y \otimes Z,
\end{tikzcd}$$

where the rows are the identity morphisms by the triangle axiom (1.2.1), the left square commutes by the definition of f, the right square commutes by naturality of θ, and the central square commutes since θ is a morphism in $\mathcal{C}'$.

Next, if $L(f) = L(g)$ for some morphisms f, g in $\mathcal{C}$ then, in particular $f \otimes \mathrm{Id}_{\mathbf{1}} = g \otimes \mathrm{Id}_{\mathbf{1}}$ so that $f = g$. Thus L is faithful.

Finally, we define a monoidal functor structure $J_{X,Y} : L(X) \circ L(Y) \xrightarrow{\sim} L(X \otimes Y)$ on L by

$$\begin{aligned} J_{X,Y} = a^{-1}_{X,Y,\bullet} &: X \otimes (Y \otimes \bullet), ((\mathrm{Id}_X \otimes a_{Y,\bullet,\bullet}) \circ a_{X,Y\otimes\bullet,\bullet}) \\ &\xrightarrow{\sim} ((X \otimes Y) \otimes \bullet,\ a_{X\otimes Y,\bullet,\bullet}). \end{aligned}$$

The diagram (1.8.2) for the latter natural isomorphism is just the pentagon diagram in $\mathcal{C}$. For the functor L the hexagon diagram (1.4.1) in the definition of a monoidal functor also reduces to the pentagon diagram in $\mathcal{C}$. The theorem is proved. $\square$

Remark 1.8.6. The nontrivial nature of MacLane's strictness theorem is demonstrated by the following instructive example, which shows that even though a monoidal category is always **equivalent** to a strict category, it need **not** be **isomorphic** to one. (By definition, an isomorphism of monoidal categories is a monoidal equivalence which is an isomorphism of categories).

Namely, let $\mathcal{C}$ be the category $\mathcal{C}_G^\omega$. If ω is cohomologically nontrivial, this category is clearly not isomorphic to a strict one. However, by Maclane's strictness theorem, it is equivalent to a strict category $\mathcal{C}'$.

In fact, in this example a strict category $\mathcal{C}'$ monoidally equivalent to $\mathcal{C}$ can be constructed quite explicitly, as follows. Let $\widetilde{G}$ be another group with a surjective homomorphism $f : \widetilde{G} \to G$ such that the 3-cocycle $f^*\omega$ is cohomologically trivial. Such $\widetilde{G}$ always exists, e.g., a free group (recall that the cohomology of a free group in degrees higher than 1 is trivial, see [Br]). Let $\mathcal{C}'$ be the category whose objects δ_g are labeled by elements of $\widetilde{G}$, $\mathsf{Hom}(\delta_g, \delta_h) = A$ if g, h have the same image in G, and $\mathsf{Hom}(\delta_g, \delta_h) = \emptyset$ otherwise. This category has an obvious tensor product, and a monoidal structure defined by the 3-cocycle $f^*\omega$. We have an obvious monoidal functor $F : \mathcal{C}' \to \mathcal{C}$ defined by the homomorphism $\widetilde{G} \to G$, and it is an equivalence, even though not an isomorphism. However, since the cocycle $f^*\omega$ is cohomologically trivial, the category $\mathcal{C}'$ is isomorphic to the same category with the trivial associativity isomorphism, which is strict.

Remark 1.8.7. [6] A category is called *skeletal* if it has only one object in each isomorphism class. The axiom of choice implies that any category is equivalent to a skeletal one. Also, by MacLane's theorem, any monoidal category is monoidally equivalent to a strict one. However, Remark 1.8.6 shows that a monoidal category need not be monoidally equivalent to a category which is skeletal and strict at the same time. Indeed, as we have seen, to make a monoidal category strict, it may be necessary to add new objects to it (which are isomorphic, but not equal to already existing ones). In fact, the desire to avoid adding such objects is the reason why we sometimes use nontrivial associativity isomorphisms, even though MacLane's strictness theorem tells us we don't have to. This also makes precise the sense in which the categories Sets, Vec, Vec_G, are "more strict" than the category Vec_G^ω for cohomologically nontrivial ω. Namely, the first three categories are monoidally equivalent to strict skeletal categories $\overline{\mathrm{Sets}}$, $\overline{\mathrm{Vec}}$, $\overline{\mathrm{Vec}_G}$, while the category Vec_G^ω is not monoidally equivalent to a strict skeletal category.

Exercise 1.8.8. Show that any monoidal category $\mathcal{C}$ is monoidally equivalent to a skeletal monoidal category $\overline{\mathcal{C}}$. Moreover, $\overline{\mathcal{C}}$ can be chosen in such a way that $l_X, r_X = \mathrm{Id}_X$ for all objects $X \in \overline{\mathcal{C}}$.

Hint. Without loss of generality one can assume that $\mathbf{1} \otimes X = X \otimes \mathbf{1} = X$ and $l_X, r_X = \mathrm{Id}_X$ for all objects $X \in \mathcal{C}$. Now in every isomorphism class i of objects of $\mathcal{C}$ fix a representative X_i, so that $X_1 = \mathbf{1}$, and for any two classes i, j fix an isomorphism $\mu_{ij} : X_i \otimes X_j \to X_{i \cdot j}$, so that $\mu_{i1} = \mu_{1i} = \mathrm{Id}_{X_i}$. Let $\overline{\mathcal{C}}$ be the full subcategory of $\mathcal{C}$ consisting of the objects X_i, with tensor product defined by $X_i \overline{\otimes} X_j = X_{i \cdot j}$, and

[6] This remark is borrowed from the paper [Kup2].

with all the structure transported using the isomorphisms μ_{ij}. Then $\overline{\mathcal{C}}$ is the required skeletal category, monoidally equivalent to $\mathcal{C}$.

18.769 Topics in Lie Theory: Tensor Categories
Spring 2009