CH. II, EXERCISES AND FURTHER RESULTS

A. On the Geometry of Lie Groups

1. Let G be a Lie group, $L(x)$ and $R(x)$, respectively the left translation $g \to xg$, and the right translation $g \to gx$. Prove:

(i) $\mathrm{Ad}(x) = dR(x^{-1})_x \circ dL(x)_e = dL(x)_{x^{-1}} \circ dR(x^{-1})_e$.

(ii) If J is the map $g \to g^{-1}$ then

$$dJ_x = -dL(x^{-1})_e \circ dR(x^{-1})_x = -dR(x^{-1})_e \circ dL(x^{-1})_x.$$

(iii) If Φ is the mapping $(g, h) \to gh$ of $G \times G$ into G, then if $X \in G_g$, $Y \in G_h$,

$$d\Phi_{(g,h)}(X, Y) = dL(g)_h(Y) + dR(h)_g(X).$$

2. Let $\gamma(t)$ $(t \in \boldsymbol{R})$ be a one-parameter subgroup of a Lie group. Assume that γ intersects itself. Then γ is a "closed" one-parameter subgroup, that is, there exists a number $L > 0$ such that $\gamma(t + L) = \gamma(t)$ for all $t \in \boldsymbol{R}$.

3. Let $\gamma(t)$, $\delta(t)$ $(t \in \boldsymbol{R})$ be two one-parameter subgroups of a Lie group. If $\gamma(L) = \delta(L)$ for some $L > 0$, then the curve $\sigma(t) = \gamma(t)\,\delta(-t)$ $(0 \leqslant t \leqslant L)$ is smooth at e, that is, $\dot{\sigma}(e) = \dot{\sigma}(L)$ (Goto and Jakobsen [ ]).

4. Let G be a locally compact group, H a closed subgroup. Prove that the space G/H is complete in any G-invariant metric.

5. Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. Let B be a nondegenerate symmetric bilinear form on $\mathfrak{g} \times \mathfrak{g}$. Then there exists a unique left invariant pseudo-Riemannian structure Q on G such that $Q_e = B$. Show, using Prop. 1.4 and (2), §9, Chapter I, that the following conditions are equivalent:

(i) The geodesics through e are the one-parameter subgroups.

(ii) $B(X, [X, Y]) = 0$, for all $X, Y \in \mathfrak{g}$.

(iii) $B(X, [Y, Z]) = B([X, Y], Z)$ for all $X, Y, Z \in \mathfrak{g}$.

(iv) Q is invariant under all right translations on G.

(v) Q is invariant under the mapping $g \to g^{-1}$ of G onto itself.

6. Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. Then there exists a unique affine connection ∇ on G invariant under all left and right translations and under the map $J: g \to g^{-1}$. Let $X, Y \in \mathfrak{g}$. Prove that:

(i) The parallel translate of X along the curve $\gamma(t) = \exp tY$ $(0 \leqslant t \leqslant 1)$ is given by

$$dL(\exp \tfrac{1}{2}Y)\, dR(\exp \tfrac{1}{2}Y)X.$$

(ii) $\nabla_{\tilde{X}}(\tilde{Y}) = \frac{1}{2}[\tilde{X}, \tilde{Y}]$ where $\tilde{X}$ and $\tilde{Y}$ are the left invariant vector fields with $\tilde{X}_e = X$, $\tilde{Y}_e = Y$.

(iii) The geodesics are the translates of one-parameter subgroups.

B. The Exponential Mapping

1. Let $\boldsymbol{SL}(2, \boldsymbol{R})$ denote the group of all real 2×2 matrices with determinant 1. Its Lie algebra $\mathfrak{sl}(2, \boldsymbol{R})$ consists of all real 2×2 matrices of trace 0.

(i) Let $X \in \mathfrak{sl}(2, \boldsymbol{R})$, $I =$ unit matrix. Show that

$$e^X = \cosh(-\det X)^{1/2} I + \frac{\sinh(-\det X)^{1/2}}{(-\det X)^{1/2}} X \qquad \text{if } \det X < 0$$

$$e^X = \cos(\det X)^{1/2} I + \frac{\sin(\det X)^{1/2}}{(\det X)^{1/2}} X \qquad \text{if } \det X > 0$$

$$e^X = I + X \qquad \text{if } \det X = 0.$$

(ii) Let us consider one-parameter subgroups the same if they have proportional tangent vectors at e. Then the matrix

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \in \boldsymbol{SL}(2, \boldsymbol{R}) \qquad (\lambda \neq 1)$$

lies on exactly one one-parameter subgroup if $\lambda > 0$, on infinitely many one-parameter subgroups if $\lambda = -1$ and one no one-parameter subgroup if $\lambda < 0$, $\lambda \neq -1$.

3. The Lie group $\boldsymbol{GL}(n, \boldsymbol{C})$ has Lie algebra $\mathfrak{gl}(n, \boldsymbol{C})$ and the mapping

$$\exp : \mathfrak{gl}(n, \boldsymbol{C}) \to \boldsymbol{GL}(n, \boldsymbol{C})$$

is surjective. (Use the Jordan canonical form).

4. Let G denote the subgroup of $\boldsymbol{GL}(n, \boldsymbol{R})$ given by

$$\begin{pmatrix} \cos\gamma & \sin\gamma & 0 & \alpha \\ -\sin\gamma & \cos\gamma & 0 & \beta \\ 0 & 0 & 1 & \gamma \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (\alpha, \beta, \gamma \in \boldsymbol{R})$$

Describe its Lie algebra $\mathfrak{g} \subset \mathfrak{gl}(n, \boldsymbol{R})$, show that $\mathfrak{g}$ is solvable, but that the mapping $\exp : \mathfrak{g} \to G$ is neither injective nor surjective.

5. Using the exponential mapping show that each Lie group G contains a neighborhood of e containing no subgroup $\neq \{e\}$. ("A Lie group has no small subgroups.")

C. Subgroups and Transformation Groups

1. Verify the description of the Lie algebras of the various subgroups of $GL(n, C)$ listed in Chapter X, §2.

2. Show that a commutative connected Lie group is isomorphic to a product group of the form $R^n \times T^m$ where T^m is an m-dimensional torus. Deduce that a one-parameter subgroup γ of a Lie group H is either closed or has compact closure.

3. Let $H \subset G$ be connected Lie groups. Suppose the identity mapping $I : H \to G$ is continuous. Then H is a Lie subgroup of G.

4. (The analytic structure of G/H) With the notation prior to Theorem 4.2 let $g, g' \in G$ and consider the two homeomorphisms

$$\psi_g : g \exp(x_1 X_1 + \dots + x_r X_r) \cdot p_0 \to (x_1, \dots, x_r) \qquad \text{of } \; g \cdot N_0 \text{ into } R^r;$$

$$\psi_{g'} : g' \exp(y_1 X_1 + \dots + y_r X_r) \cdot p_0 \to (y_1, \dots, y_r) \qquad \text{of } \; g' \cdot N_0 \text{ into } R^r.$$

Prove that the mapping $\psi_{g'} \circ \psi_g^{-1}$ is an analytic mapping of

$$\psi_g(g \cdot N_0 \cap g' \cdot N_0)$$

onto

$$\psi_{g'}(g \cdot N_0 \cap g' \cdot N_0).$$

5. Let G be a Lie transformation group of a manifold M. Then each orbit $G \cdot p$ is a submanifold of M, diffeomorphic to G/G_p. (Proceed as in the proof of Prop. 4.3.)

6. Let G be a locally connected topological group. Suppose the identity component G_0 has an analytic structure compatible with the topology in which it is a Lie group. Show that G has the same property. (Hint: Use Theorem 2.6.)

This shows that the definition of a Lie group adopted here is equivalent to that of Chevalley | Theory of Lie Groups I .

7*. Suppose an abstract subgroup H of a connected Lie group G has a manifold structure in which it is a submanifold of G with at most countably many components. Then H is a Lie subgroup of G. (cf. Freudenthal [4]; see also Kobayashi and Nomizu [1], I, p. 275 or F. Warner [1], p. 95, and Chevalley [2], p. 96).

8. Let G be a connected Lie group, $H \subset G$ a closed subgroup. The action of G on the manifold $M = G/H$ is called *imprimitive* if there exists a connected submanifold N of M ($0 < \dim N < \dim M$) such that for each $g \in G$ either $g \cdot N = N$ or $g \cdot N \cap N = \emptyset$. Show that this is equivalent to the existence of a Lie subgroup L, $H \subset L \subset G$, such that $\dim H < \dim L < \dim G$.

9*. Let G be a Lie transformation group of a manifold M, M/G the orbit space topologized by the finest topology for which the natural mapping $\pi : M \to M/G$ is continuous. Let

$$D = \{(p, q) \in M \times M : p = g \cdot q \text{ for some } g \in G\}.$$

Prove that:

(i) M/G is a Hausdorff space if and only if the subset $D \subset M \times M$ is closed.

(ii) There exists a differentiable structure on the topological space M/G such that $\pi : M \to M/G$ is a submersion if and only if the topological subspace $D \subset M \times M$ is a closed submanifold.

In this case the differentiable structure is unique and all the G-orbits in M have the same dimension (see, e.g., Dieudonné [2], Chapitre XVI).

D. Closed Subgroups

1. Let Γ be a discrete subgroup of R^2 such that R^2/Γ is compact. Show that an analytic subgroup of R^2 is always closed but that its image in R^2/Γ under the natural mapping is not necessarily closed.

2. Let $\mathfrak{g}$ be a Lie algebra such that $\mathrm{Int}(\mathfrak{g})$ has compact closure in $GL(\mathfrak{g})$. Then $\mathrm{Int}(\mathfrak{g})$ is compact. (Hint: Repeat the proof of Prop. 6.6 and use Prop. 6.6(i).)

3. Let G denote the five-dimensional manifold $C \times C \times R$ with multiplication defined as follows (van Est [1], Hochschild [1]):

$$(c_1, c_2, r)(c_1', c_2', r') = (c_1 + e^{2\pi i r}c_1', c_2 + e^{2\pi i h r}c_2', r + r'),$$

where h is a fixed irrational number and $c_1, c_2, c_1', c_2' \in C$, $r, r' \in R$. Then G is a Lie group.

(i) Let $s, t \in R$ and define the mapping $\alpha_{s,t} : G \to G$ by $\alpha_{s,t}(c_1, c_2, r) = (e^{2\pi i s}c_1, e^{2\pi i t}c_2, r)$. Show that $\alpha_{s,t}$ is an analytic isomorphism.

(ii) If $t = hs + hn$ where n is an integer, then $\alpha_{s,t}$ coincides with the inner automorphism

$$(c_1, c_2, r) \to (0, 0, s + n)(c_1, c_2, r)(0, 0, s + n)^{-1}.$$

(iii) Let $\mathfrak{g}$ denote the Lie algebra of G and let $A_{s,t}$ denote the automorphism $d\alpha_{s,t}$ of $\mathfrak{g}$. If $s_n \to s_0$, $t_n \to t_0$ then $A_{s_n,t_n} \to A_{s_0,t_0}$ in $\mathrm{Aut}(\mathfrak{g})$.

(iv) Show that $A_{0,1/3} \notin \mathrm{Int}(\mathfrak{g})$. Deduce from (iii) that $\mathrm{Int}(\mathfrak{g})$ is not closed in $\mathrm{Aut}(\mathfrak{g})$.

4*. Let G be a connected Lie group and H an analytic subgroup. Let $\mathfrak{g}$ and $\mathfrak{h}$ denote the corresponding Lie algebras.

(i) Assume G simply connected. If $\mathfrak{h}$ is an ideal in $\mathfrak{g}$ then H is closed in G (Chevalley [2], p. 127).

(ii) Assume G simply connected. If $\mathfrak{h}$ is semisimple then H is closed in G (Mostow [2], p. 615).

(iii) Assume G compact. If $\mathfrak{h}$ is semisimple then H is closed in G (Mostow [2], p. 615).

(iv) Assume $G = GL(n, C)$. If $\mathfrak{h}$ is semisimple then H is closed in G (Goto [1], Yosida [1]).

(v) Suppose H is not closed in G. Then there exists a one-parameter subgroup γ of H whose closure (in G) is not contained in H (Goto [1]).

(vi) H is closed if $\exp \mathfrak{h}$ is closed. This follows from (v).

(vii) Assume G solvable and simply connected. Then H is closed and simply connected (Chevalley [8]).

(viii) Suppose $G = SO(n)$ and that H acts irreducibly on R^n. Then H is closed in G (Borel and Lichnèrowicz [2], Kobayashi and Nomizu [1], I, p. 277).

E. Invariant Differential Forms

1. Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. Let $(X_i)_{1 \leqslant i \leqslant n}$ be a basis of $\mathfrak{g}$, $\tilde{X}_i$ $(1 \leqslant i \leqslant n)$ the corresponding left invariant vector fields, and ω_j $(1 \leqslant j \leqslant n)$ the dual forms given by $\omega_j(\tilde{X}_i) = \delta_{ij}$. From (1), §7 or Exercise C4, Chapter I deduce the formula (cf. Koszul [4])

$$2\, d\omega = \sum_{k=1}^{n} \omega_k \wedge \theta(\tilde{X}_k)\omega \qquad (\omega \text{ left invariant})$$

where $\theta(\tilde{X}_k)$ is the Lie derivative (Exercise B.1, Chapter I). Show that if $\omega = \omega_i$, this formula reduces to the Maurer-Cartan equations (3), §7.

2. Prove that for the orthogonal group $\boldsymbol{O}(n)$ the matrix of 1-forms $\Omega = g^{-1}\, dg$ $(g \in \boldsymbol{O}(n))$ satisfies

$$d\Omega + \Omega \wedge \Omega = 0, \qquad \Omega + {}^t\Omega = 0,$$

${}^tA$ denoting the transpose of a matrix A. Generalize these relations to $\boldsymbol{U}(n)$ and $\boldsymbol{Sp}(n)$.

3. Using the method of Exercise E2 show that the group of matrices

$$g = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \qquad (x, y, z \in \boldsymbol{R})$$

has a basis of left invariant 1-forms given by

$$\omega_1 = dx, \qquad \omega_2 = dy, \qquad \omega_3 = dz - x\, dy$$

and that the Maurer-Cartan equations are

$$d\omega_1 = 0, \qquad d\omega_2 = 0, \qquad d\omega_3 = -\omega_1 \wedge \omega_2.$$

F. Invariant Measures

1. Let G be a Lie group and H a closed subgroup. Then

(i) If H is compact, G/H has an invariant measure.

(ii) If G is unimodular and H normal, then H is unimodular.

(iii) If G/H has a finite invariant measure and if H is unimodular, then G is unimodular.

2. For the group $O(2)$ the element $g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ satisfies $\mathrm{Ad}(g) = -I$.

3. Let G be a connected Lie group with Lie algebra $\mathfrak{g}$, and $H \subset G$ a closed analytic subgroup with Lie algebra $\mathfrak{h} \subset \mathfrak{g}$. Let $X_1, \ldots, X_n$ be a basis of $\mathfrak{g}$ such that $X_{r+1}, \ldots, X_n$ span $\mathfrak{h}$ and put

$$\mathfrak{m} = \mathbf{R}X_1 + \cdots + \mathbf{R}X_r.$$

Let c_{ij}^k be determined by $[X_i, X_j] = \sum_k c_{ij}^k X_k$.

(i) G is unimodular if and only if

$$\mathrm{Tr}_{\mathfrak{g}}(\mathrm{ad}\, X) = 0 \qquad \text{for} \quad X \in \mathfrak{g},$$

or, equivalently,

$$\sum_{k=1}^{n} c_{ik}^k = 0 \qquad \text{for} \quad \text{each } i, \quad 1 \leq i \leq n.$$

(ii) The space G/H has an invariant measure if and only if

$$\mathrm{Tr}(\mathrm{ad}_{\mathfrak{g}}(T)) = \mathrm{Tr}(\mathrm{ad}_{\mathfrak{h}}(T)) \qquad \text{for} \quad T \in \mathfrak{h},$$

or, equivalently,

$$\sum_{\alpha=1}^{r} c_{i\alpha}^{\alpha} = 0 \qquad \text{for} \quad r+1 \leq i \leq n$$

(Chern [1942]).

4. Show that the group $\mathbf{M}(n)$ of isometries of $\mathbf{R}^n$ is isomorphic to the group of matrices

$$g_{k,x} = \begin{pmatrix} & & & x_1 \\ & k & & \vdots \\ & & & x_n \\ 0 & \cdots & 0 & 1 \end{pmatrix},$$

where $k \in K = \mathbf{O}(n)$ and $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$. A Haar measure dg on $\mathbf{M}(n)$ is then given by

$$\int_G f(g)\, dg = \int_{K \times \mathbf{R}^n} f(g_{k,x})\, dk\, dx, \qquad f \in C_c(\mathbf{M}(n)),$$

where dk is a Haar measure on K.

5. A biinvariant measure on the group $G = \boldsymbol{GL}(n, \boldsymbol{R})$ of nonsingular matrices $X = (x_{ij})$ is given by

$$f \to \int_G f(X)|\det X|^{-n} \prod_{i,j} dx_{ij}.$$

6. A biinvariant measure on the unimodular group $G = \boldsymbol{SL}(n, \boldsymbol{R})$ is given by

$$f \to \int_{G'} f(X)|\det X_{11}|^{-1} \prod_{(i,j)\neq(1,1)} dx_{ij}.$$

Here $X = (x_{ij})$, X_{ij} is the (i, j)-cofactor in X, and the x_{ij} (except for x_{11}) are taken as independent variables on the set G' given by $\det X_{11} \neq 0$.

7. Let $\boldsymbol{T}(n, \boldsymbol{R})$ denote the group of all $g \in \boldsymbol{GL}(n, \boldsymbol{R})$ which are upper triangular. A left-invariant measure on $\boldsymbol{T}(n, \boldsymbol{R})$ is given by

$$f \to \int_{\boldsymbol{T}(n,\boldsymbol{R})} f(t) t_{11}^{-n} t_{22}^{1-n} \cdots t_{nn}^{-1} \prod_{i \leq j} dt_{ij}$$

and a right-invariant measure by

$$f \to \int_{\boldsymbol{T}(n,\boldsymbol{R})} f(t) t_{11}^{-1} t_{22}^{-2} \cdots t_{nn}^{-n} \prod_{i \leq j} dt_{ij}.$$

G. Compact Real Forms and Complete Reducibility

1. Show that the function f (after Theorem 6.3) has minimum value when the structural constants are real if and only if the real span of the basis vectors is a compact real form. Show that the minimum value then equals n.

3. A representation ρ of a Lie algebra $\mathfrak{g}$ (resp. a group G) on a finite-dimensional vector space V is called *semisimple* (or *completely reducible*) if each subspace of V invariant under $\rho(\mathfrak{g})$ (resp. $\rho(G)$) has a complementary invariant subspace.

(i) Any finite-dimensional representation of a compact topological group on a real or complex vector space V is semisimple.

(ii) (Weyl's unitary trick) Using a compact real form prove that any finite-dimensional representation π of a real semisimple Lie algebra $\mathfrak{g}$ on a real or complex vector space V is semisimple.