## 9. Calogero-Moser spaces

9.1. **Hamiltonian reduction along an orbit.** Let $\mathcal{M}$ be an affine algebraic variety and $G$ a reductive algebraic group. Suppose $\mathcal{M}$ is Poisson and the action of $G$ preserves the Poisson structure. Let $\mathfrak{g}$ be the Lie algebra of $G$ and $\mathfrak{g}^*$ the dual of $\mathfrak{g}$. Let $\mu : \mathcal{M} \to \mathfrak{g}^*$ be a moment map for this action (we assume it exists). It induces a map $\mu^* : S\mathfrak{g} \to \mathbb{C}[\mathcal{M}]$.

Let $\mathcal{O}$ be a closed coadjoint orbit of $G$, $I_{\mathcal{O}}$ be the ideal in $S\mathfrak{g}$ corresponding to $\mathcal{O}$, and let $J_{\mathcal{O}}$ be the ideal in $\mathbb{C}[\mathcal{M}]$ generated by $\mu^*(I_{\mathcal{O}})$. Then $J_{\mathcal{O}}^G$ is a Poisson ideal in $\mathbb{C}[\mathcal{M}]^G$, and $A = \mathbb{C}[\mathcal{M}]^G/J_{\mathcal{O}}^G$ is a Poisson algebra.

Geometrically, $\mathrm{Spec}(A) = \mu^{-1}(\mathcal{O})/G$ (categorical quotient). It can also be written as $\mu^{-1}(z)/G_z$, where $z \in \mathcal{O}$ and $G_z$ is the stabilizer of $z$ in $G$.

**Definition 9.1.** The scheme $\mu^{-1}(\mathcal{O})/G$ is called *the Hamiltonian reduction of $\mathcal{M}$ with respect to $G$ along* $\mathcal{O}$. We will denote by $R(\mathcal{M}, G, \mathcal{O})$.

The following proposition is standard.

**Proposition 9.2.** *If $\mathcal{M}$ is a symplectic variety and the action of $G$ on $\mu^{-1}(\mathcal{O})$ is free, then $R(\mathcal{M}, G, \mathcal{O})$ is a symplectic variety, of dimension* $\dim(\mathcal{M}) - 2\dim(G) + \dim(\mathcal{O})$.

9.2. **The Calogero-Moser space.** Let $\mathcal{M} = T^*\mathrm{M}at_n(\mathbb{C})$, and $G = \mathrm{PGL}_n(\mathbb{C})$ (so $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$). Using the trace form we can identify $\mathfrak{g}^*$ with $\mathfrak{g}$, and $\mathcal{M}$ with $\mathrm{M}at_n(\mathbb{C}) \oplus \mathrm{M}at_n(\mathbb{C})$. Then a moment map is given by the formula $\mu(X, Y) = [X, Y]$, for $X, Y \in \mathrm{M}at_n(\mathbb{C})$.

Let $\mathcal{O}$ be the orbit of the matrix $\mathrm{diag}(-1, -1, \dots, -1, n-1)$, i.e. the set of traceless matrices $T$ such that $T + 1$ has rank 1.

**Definition 9.3** (Kazhdan, Kostant, Sternberg, [KKS])**.** The scheme $\mathcal{C}_n := R(\mathcal{M}, G, \mathcal{O})$ is called *the Calogero-Moser space.*

**Proposition 9.4.** *The action of $G$ on $\mu^{-1}(\mathcal{O})$ is free, and thus (by Proposition 9.2) $\mathcal{C}_n$ is a smooth symplectic variety (of dimension $2n$).*

*Proof.* It suffices to show that if $X, Y$ are such that $XY - YX + 1$ has rank 1, then $(X, Y)$ is an irreducible set of matrices. Indeed, in this case, by Schur's lemma, if $B \in \mathrm{GL}_n$ is such that $BX = XB$ and $BY = YB$ then $B$ is a scalar, so the stabilizer of $(X, Y)$ in $\mathrm{PGL}_n$ is trivial.

To show this, assume that $\mathcal{W} \neq 0$ is an invariant subspace of $X, Y$. In this case, the eigenvalues of $[X, Y]$ on $\mathcal{W}$ are a subcollection of the collection of $n-1$ copies of $-1$ and one copy of $n-1$. The sum of the elements of this subcollection must be zero, since it is the trace of $[X, Y]$ on $\mathcal{W}$. But then the subcollection must be the entire collection, so $\mathcal{W} = \mathbb{C}^n$, as desired. □

Thus, $\mathcal{C}_n$ is the space of conjugacy classes of pairs of $n \times n$ matrices $(X, Y)$ such that the matrix $XY - YX + 1$ has rank 1.

In fact, one also has the following more complicated theorem.

**Theorem 9.5** (G. Wilson, [Wi])**.** *The Calogero-Moser space is connected.*

We will give a proof of this theorem later, in Subsection 9.4.

9.3. **The Calogero-Moser integrable system.** Let $\mathcal{M}$ be a symplectic variety, and let $H_1, \ldots, H_n$ be regular functions on $\mathcal{M}$ such that $\{H_i, H_j\} = 0$ and $H_i$'s are algebraically independent everywhere. Assume that $\mathcal{M}$ carries a symplectic action of a reductive algebraic group $G$ with moment map $\mu : \mathcal{M} \to \mathfrak{g}^*$, which preserves the functions $H_i$, and let $\mathcal{O}$ be a coadjoint orbit of $G$. Assume that $G$ acts freely on $\mu^{-1}(\mathcal{O})$, and so the Calogero-Moser space $R(\mathcal{M}, G, \mathcal{O})$ is symplectic. The functions $H_i$ descend to $R(\mathcal{M}, G, \mathcal{O})$. If they are still algebraically independent and $n = \dim R(\mathcal{M}, G, \mathcal{O})/2$, then we get an integrable system on $R(\mathcal{M}, G, \mathcal{O})$.

A vivid example of this is the Kazhdan-Kostant-Sternberg construction of the Calogero-Moser system. In this case $\mathcal{M} = T^*\mathrm{Mat}_n(\mathbb{C})$ (regarded as the set of pairs of matrices $(X, Y)$ as in Section 9.2), with the usual symplectic form $\omega = \mathrm{Tr}\,(\mathrm{d}Y \wedge \mathrm{d}X)$. Let $H_i = \mathrm{Tr}\,(Y^i)$, $i = 1, \ldots, n$. Let $G = \mathrm{PGL}_n(\mathbb{C})$ act on $\mathcal{M}$ by conjugation, and let $\mathcal{O}$ be the coadjoint orbit of $G$ considered in Subsection 9.2. Then the system $H_1, \ldots, H_n$ descends to a system of functions in involution on $R(\mathcal{M}, G, \mathcal{O})$, which is the Calogero-Moser space $\mathcal{C}_n$. Since this space is $2n$-dimensional, $H_1, \ldots, H_n$ form an integrable system on $\mathcal{C}_n$. It is called *the (rational) Calogero-Moser system.*

The Calogero-Moser flow is, by definition, the Hamiltonian flow on $\mathcal{C}_n$ defined by the Hamiltonian $H = H_2 = \mathrm{Tr}\,(Y^2)$. Thus this flow is integrable, in the sense that it can be included in an integrable system. In particular, its solutions can be found in quadratures using the inductive procedure of reduction of order. However (as often happens with systems obtained by reduction), solutions can also be found by a much simpler procedure, since they can be found already on the "non-reduced" space $\mathcal{M}$: indeed, on $\mathcal{M}$ the Calogero-Moser flow is just the motion of a free particle in the space of matrices, so it has the form $g_t(X, Y) = (X + 2Yt, Y)$. The same formula is valid on $\mathcal{C}_n$. In fact, we can use the same method to compute the flows corresponding to all the Hamiltonians $H_i = \mathrm{Tr}\,(Y^i)$, $i \in \mathbb{N}$: these flows are given by the formulas

$$g_t^{(i)}(X, Y) = (X + iY^{i-1}t, Y).$$

Let us write the Calogero-Moser system explicitly in coordinates. To do so, we first need to introduce local coordinates on the Calogero-Moser space $\mathcal{C}_n$.

To this end, let us restrict our attention to the open set $U_n \subset \mathcal{C}_n$ which consists of conjugacy classes of those pairs $(X, Y)$ for which the matrix $X$ is diagonalizable, with distinct eigenvalues; by Wilson's Theorem 9.5, this open set is dense in $\mathcal{C}_n$.

A point $P \in U_n$ may be represented by a pair $(X, Y)$ such that $X = \mathrm{diag}(x_1, \ldots, x_n)$, $x_i \neq x_j$. In this case, the entries of $T := XY - YX$ are $(x_i - x_j)y_{ij}$. In particular, the diagonal entries are zero. Since the matrix $T + 1$ has rank 1, its entries $\kappa_{ij}$ have the form $a_i b_j$ for some numbers $a_i, b_j$. On the other hand, $\kappa_{ii} = 1$, so $b_j = a_j^{-1}$ and hence $\kappa_{ij} = a_i a_j^{-1}$. By conjugating $(X, Y)$ by the matrix $\mathrm{diag}(a_1, \ldots, a_n)$, we can reduce to the situation when $a_i = 1$, so $\kappa_{ij} = 1$. Hence the matrix $T$ has entries $1 - \delta_{ij}$ (zeros on the diagonal, ones off the diagonal). Moreover, the representative of $P$ with diagonal $X$ and $T$ as above is unique up to the action of the symmetric group $\mathfrak{S}_n$. Finally, we have $(x_i - x_j)y_{ij} = 1$ for $i \neq j$, so the entries of the matrix $Y$ are $y_{ij} = 1/(x_i - x_j)$ if $i \neq j$. On the other hand, the diagonal entries $y_{ii}$ of $Y$ are unconstrained. Thus we have obtained the following result.

**Proposition 9.6.** *Let $\mathbb{C}^n_{\mathrm{reg}}$ be the open set of $(x_1, \ldots, x_n) \in \mathbb{C}^n$ such that $x_i \neq x_j$ for $i \neq j$. Then there exists an isomorphism of algebraic varieties $\xi : T^*(\mathbb{C}^n_{\mathrm{reg}}/\mathfrak{S}_n) \to U_n$ given by the* *formula*

$$(x_1, \dots, x_n, p_1, \dots, p_n) \mapsto (X, Y),$$

*where* $X = \mathrm{diag}(x_1, \dots, x_n)$, *and* $Y = Y(\mathbf{x}, \mathbf{p}) := (y_{ij})$,

$$y_{ij} = \frac{1}{x_i - x_j}, i \neq j, \; y_{ii} = p_i.$$

In fact, we have a stronger result:

**Proposition 9.7.** *$\xi$ is an isomorphism of symplectic varieties (where the cotangent bundle is equipped with the usual symplectic structure).*

For the proof of Proposition 9.7, we will need the following general and important but easy theorem.

**Theorem 9.8** (The necklace bracket formula)**.** *Let $a_1, \dots, a_r$ and $b_1, \dots, b_s$ be either $X$ or $Y$. Then on $\mathcal{M}$ we have*

$$\{\mathrm{Tr}\,(a_1 \cdots a_r), \mathrm{Tr}\,(b_1 \cdots b_s)\} = \sum_{(i,j):a_i=Y,b_j=X} \mathrm{Tr}\,(a_{i+1} \cdots a_r a_1 \cdots a_{i-1} b_{j+1} \cdots b_s b_1 \cdots b_{j-1}) - \sum_{(i,j):a_i=X,b_j=Y} \mathrm{Tr}\,(a_{i+1} \cdots a_r a_1 \cdots a_{i-1} b_{j+1} \cdots b_s b_1 \cdots b_{j-1}).$$

*Proof of Proposition 9.7.* Let $a_k = \mathrm{Tr}\,(X^k)$, $b_k = \mathrm{Tr}\,(X^k Y)$. It is easy to check using the necklace bracket formula that on $\mathcal{M}$ we have

$$\{a_m, a_k\} = 0, \quad \{b_m, a_k\} = k a_{m+k-1}, \quad \{b_m, b_k\} = (k-m) b_{m+k-1}.$$

On the other hand, $\xi^* a_k = \sum x_i^k$, $\xi^* b_k = \sum x_i^k p_i$. Thus we see that

$$\{f, g\} = \{\xi^* f, \xi^* g\},$$

where $f, g$ are either $a_k$ or $b_k$. But the functions $a_k, b_k$, $k = 0, \dots, n-1$, form a local coordinate system near a generic point of $U_n$, so we are done. □

Now let us write the Hamiltonian of the Calogero-Moser system in coordinates. It has the form

$$H = \mathrm{Tr}\,(Y(\mathbf{x}, \mathbf{p})^2) = \sum_i p_i^2 - \sum_{i \neq j} \frac{1}{(x_i - x_j)^2}. \tag{9.1}$$

Thus the Calogero-Moser Hamiltonian describes the motion of a system of $n$ particles on the line with interaction potential $-1/x^2$, which we considered in Section 2.

Now we finally see the usefulness of the Hamiltonian reduction procedure. The point is that it is not clear at all from formula (9.1) why the Calogero-Moser Hamiltonian should be completely integrable. However, our reduction procedure implies the complete integrability of $H$, and gives an explicit formula for the first integrals: [7]

$$H_i = \mathrm{Tr}\,(Y(\mathbf{x}, \mathbf{p})^i).$$

Moreover, this procedure immediately gives us an explicit solution of the system. Namely, assume that $\mathbf{x}(t), \mathbf{p}(t)$ is the solution with initial condition $\mathbf{x}(0), \mathbf{p}(0)$. Let $(X_0, Y_0) =$

[7] Thus, for type $A$ we have two methods of proving the integrability of the Calogero-Moser system - one using Dunkl operators and one using Hamiltonian reduction.

$\xi(\mathbf{x}(0), \mathbf{p}(0))$. Then $x_i(t)$ are the eigenvalues of the matrix $X_t := X_0 + 2tY_0$, and $p_i(t) = x_i'(t)/2$.

9.4. **Proof of Wilson's theorem.** Let us now give a proof of Theorem 9.5.

We have already shown that all components of $\mathcal{C}_n$ are smooth and have dimension $2n$. Also, we know that there is at least one component (the closure of $U_n$), and that the other components, if they exist, do not contain pairs $(X, Y)$ in which $X$ is regular semisimple. This means that these components are contained in the hypersurface $\Delta(X) = 0$, where $\Delta(X)$ stands for the discriminant of $X$ (i.e., $\Delta(X) := \prod_{i \neq j}(x_i - x_j)$, where $x_i$ are the eigenvalues of $X$).

Thus, to show that such additional components don't in fact exist, it suffices to show that the dimension of the subscheme $\mathcal{C}_n(0)$ cut out in $\mathcal{C}_n$ by the equation $\Delta(X) = 0$ is $2n - 1$.

To do so, first notice that the condition $\mathrm{rank}\,([X, Y] + 1) = 1$ is equivalent to the equation $\wedge^2([X, Y] + 1) = 0$; thus, the latter can be used as the equation defining $\mathcal{C}_n$ inside $T^*\mathrm{Mat}_n/PGL_n$.

Define $\mathcal{C}_n^0 := \mathrm{Spec}(\mathrm{gr}\mathcal{O}(\mathcal{C}_n))$ to be the degeneration of $\mathcal{C}_n$ (we use the filtration on $\mathcal{O}(\mathcal{C}_n)$ defined by $\deg(X) = 0$, $\deg(Y) = 1$). Then $\mathcal{C}_n^0$ is a closed subscheme in the scheme $\widetilde{\mathcal{C}}_n^0$ cut out by the equations $\wedge^2([X, Y]) = 0$ in $T^*\mathrm{M}at_n/\mathrm{PGL}_n$.

Let $(\widetilde{\mathcal{C}}_n^0)_{\mathrm{red}}$ be the reduced part of $\widetilde{\mathcal{C}}_n^0$. Then $(\widetilde{\mathcal{C}}_n^0)_{red}$ coincides with the categorical quotient $\{(X, Y)|\mathrm{rank}\,([X, Y]) \leq 1\}/\mathrm{PGL}_n$.

Our proof is based on the following proposition.

**Proposition 9.9.** *The categorical quotient* $\{(X, Y)|\mathrm{rank}\,([X, Y]) \leq 1\}/\mathrm{PGL}_n$ *coincides with the categorical quotient* $\{(X, Y)|[X, Y] = 0\}/\mathrm{PGL}_n$.

*Proof.* It is clear that $\{(X, Y)|[X, Y] = 0\}/\mathrm{PGL}_n$ is contained in $\{(X, Y)|\mathrm{rank}\,([X, Y]) \leq 1\}/\mathrm{PGL}_n$. For the proof of the opposite inclusion we need to show that any regular function on $\{(X, Y)|\mathrm{rank}\,([X, Y]) \leq 1\}/\mathrm{PGL}_n$ is completely determined by its values on the subvariety $\{(X, Y)|[X, Y] = 0\}/\mathrm{PGL}_n$, i.e. that any invariant polynomial on the set of pairs of matrices with commutator of rank at most 1 is completely determined by its values on pairs of commuting matrices. To this end, we need the following lemma from linear algebra.

**Lemma 9.10.** *If $A, B$ are square matrices such that $[A, B]$ has rank $\leq 1$, then there exists a basis in which both $A, B$ are upper triangular.*

*Proof.* Without loss of generality, we can assume $\ker A \neq 0$ (by replacing $A$ with $A - \lambda$ if needed) and that $A \neq 0$. It suffices to show that there exists a proper nonzero subspace invariant under $A, B$; then the statement will follow by induction in dimension.

Let $C = [A, B]$ and suppose $\mathrm{rank}\, C = 1$ (since the case $\mathrm{rank}\, C = 0$ is trivial). If $\ker A \subset \ker C$, then $\ker A$ is $B$-invariant: if $Av = 0$ then $ABv = BAv + Cv = 0$. Thus $\ker A$ is the required subspace. If $\ker A \not\subseteq \ker C$, then there exists a vector $v$ such that $Av = 0$ but $Cv \neq 0$. So $ABv = Cv \neq 0$. Thus $\mathrm{Im}C \subset \mathrm{Im}A$. So $\mathrm{Im}A$ is $B$-invariant: $BAv = ABv + Cv \in \mathrm{Im}A$. So $\mathrm{Im}A$ is the required subspace.

This proves the lemma. □

Now we are ready to prove Proposition 9.9. By the fundamental theorem of invariant theory, the ring of invariants of $X$ and $Y$ is generated by traces of words of $X$ and $Y$: $\mathrm{Tr}(w(X, Y))$. If $X$ and $Y$ are upper triangular with eigenvalues $x_i, y_i$, then any such trace has the form $\sum x_i^m y_i^r$, i.e. coincides with the value of the corresponding invariant on the diagonal parts $X_{\text{diag}}, Y_{\text{diag}}$ of $X$ and $Y$, which commute. The proposition is proved. □

We will also need the following proposition:

**Proposition 9.11.** *The categorical quotient* $\{(X,Y)|[X,Y]=0\}/\mathrm{PGL}_n$ *is isomorphic to* $(\mathbb{C}^n \times \mathbb{C}^n)/\mathfrak{S}_n$, *i.e. its function algebra is* $\mathbb{C}[x_1,\dots,x_n,y_1,\dots,y_n]^{\mathfrak{S}_n}$.

*Proof.* Restriction to diagonal matrices defines a homomorphism

$$\xi : \mathcal{O}(\{(X,Y)|[X,Y]=0\}/\mathrm{PGL}_n) \to \mathbb{C}[x_1,\dots,x_n,y_1,\dots,y_n]^{\mathfrak{S}_n}.$$

Since (as explained in the proof of Proposition 9.9), any invariant polynomial of entries of commuting matrices is determined by its values on diagonal matrices, this map is injective. Also, $\xi(\mathrm{Tr}\,(X^m Y^r)) = \sum x_i^m y_i^r$, where $x_i, y_i$ are the eigenvalues of $X$ and $Y$.

Now we use the following well known theorem of H. Weyl (from his book "Classical groups").

**Theorem 9.12.** *Let $B$ be an algebra over $\mathbb{C}$. Then the algebra $S^n B$ is generated by elements of the form*

$$b \otimes 1 \otimes \cdots \otimes 1 + 1 \otimes b \otimes \cdots \otimes 1 + \cdots + 1 \otimes 1 \otimes \cdots \otimes b.$$

*Proof.* Since $S^n B$ is linearly spanned by elements of the form $x \otimes \cdots \otimes x$, $x \in B$, it suffices to prove the theorem in the special case $B = \mathbb{C}[x]$. In this case, the result is simply the fact that the ring of symmetric functions is generated by power sums, which is well known. □

**Corollary 9.13.** *The ring* $\mathbb{C}[x_1,\dots,x_n,y_1,\dots,y_n]^{\mathfrak{S}_n}$ *is generated by the polynomials* $\sum x_i^m y_i^r$ *for* $m, r \geq 0$, $m + r > 0$.

*Proof.* Apply Theorem 9.12 in the case $B = \mathbb{C}[x,y]$. □

Corollary 9.13 implies that $\xi$ is surjective. Proposition 9.11 is proved. □

Now we are ready to prove Wilson's theorem. Let $\mathcal{C}_n(0)^0$ be the degeneration of $\mathcal{C}_n(0)$, i.e. the subscheme of $\mathcal{C}_n^0$ cut out by the equation $\Delta(X) = 0$. According to Propositions 9.9 and 9.11, the reduced part $(\mathcal{C}_n(0)^0)_{\text{red}}$ is contained in the hypersurface in $(\mathbb{C}^n \times \mathbb{C}^n)/\mathfrak{S}_n$ cut out by the equation $\prod_{i<j}(x_i - x_j) = 0$. This hypersurface has dimension $2n-1$, so we are done.

9.5. **The Gan-Ginzburg theorem.** Let $\mathrm{Comm}(n)$ be *the commuting scheme* defined in $T^*\mathrm{Mat}_n = \mathrm{Mat}_n \times \mathrm{Mat}_n$ by the equations $[X,Y] = 0$, $X, Y \in \mathrm{Mat}_n$. Let $G = \mathrm{PGL}_n$, and consider the categorical quotient $\mathrm{Comm}(n)/G$ (i.e., the Hamiltonian reduction $\mu^{-1}(0)/G$ of $T^*\mathrm{Mat}_n$ by the action of $G$), whose algebra of regular functions is $A = \mathbb{C}[\mathrm{Comm}(n)]^G$.

It is not known whether the commuting scheme $\mathrm{Comm}(n)$ is reduced (i.e. whether the corresponding ideal is a radical ideal); this is a well known open problem. The underlying variety is irreducible (as was shown by Gerstenhaber [Ge1]), but very singular, and has a very complicated structure. However, we have the following result.

**Theorem 9.14** (Gan, Ginzburg, [GG])**.** $\mathrm{Comm}(n)/G$ *is reduced, and isomorphic to* $\mathbb{C}^{2n}/\mathfrak{S}_n$. *Thus* $A = \mathbb{C}[x_1,\dots,x_n,y_1,\dots,y_n]^{\mathfrak{S}_n}$. *The Poisson bracket on this algebra is induced from the standard symplectic structure on* $\mathbb{C}^{2n}$.

*Sketch of the proof.* Look at the *almost commuting variety* $\mathcal{M}_n \subset \mathfrak{gl}_n \times \mathfrak{gl}_n \times \mathbb{C}^n \times (\mathbb{C}^n)^*$ defined by

$$\mathcal{M}_n = \{(X, Y, \mathbf{v}, \mathbf{f}) | [X, Y] + \mathbf{v} \otimes \mathbf{f} = 0\}.$$

Gan and Ginzburg proved the following result.

**Theorem 9.15.** *$\mathcal{M}_n$ is a complete intersection. It has $n+1$ irreducible components denoted by $\mathcal{M}_n^i$, labeled by $i = \dim \mathbb{C}\langle X, Y\rangle \mathbf{v}$. Also, $\mathcal{M}_n$ is generically reduced.*

Since $\mathcal{M}_n$ is generically reduced and is a complete intersection, by a standard result of commutative algebra it is reduced. Thus $\mathbb{C}[\mathcal{M}_n]$ has no nonzero nilpotents. This implies $\mathbb{C}[\mathcal{M}_n]^G$ has no nonzero nilpotents.

However, it is easy to show that the algebra $\mathbb{C}[\mathcal{M}_n]^G$ is isomorphic to the algebra of invariant polynomials of entries of $X$ and $Y$ modulo the "rank 1" relation $\wedge^2[X, Y] = 0$. By a scheme-theoretic version of Proposition 9.9 (proved in [EG]), the latter is isomorphic to $A$. This implies the theorem (the statement about Poisson structures is checked directly in coordinates on the open part where $X$ is regular semisimple). □

9.6. **The space $\mathsf{M}_c$ for $\mathfrak{S}_n$ and the Calogero-Moser space.** Let $\mathsf{H}_{0,c} = \mathsf{H}_{0,c}[\mathfrak{S}_n, V]$ be the symplectic reflection algebra of the symmetric group $\mathfrak{S}_n$ and space $V = \mathfrak{h} \oplus \mathfrak{h}^*$, where $\mathfrak{h} = \mathbb{C}^n$ (i.e., the rational Cherednik algebra $H_{0,c}(\mathfrak{S}_n, \mathfrak{h})$). Let $\mathsf{M}_c = \operatorname{Spec} \mathsf{B}_{0,c}[\mathfrak{S}_n, V]$ be the Calogero-Moser space defined in Section 8.5. It is a symplectic variety for $c \neq 0$.

**Theorem 9.16.** *For $c \neq 0$ the space $\mathsf{M}_c$ is isomorphic to the Calogero-Moser space $\mathcal{C}_n$ as a symplectic variety.*

*Proof.* To prove the theorem, we will first construct a map $f : \mathsf{M}_c \to \mathcal{C}_n$, and then prove that $f$ is an isomorphism.

Without loss of generality, we may assume that $c = 1$. As we have shown before, the algebra $\mathsf{H}_{0,c}$ is an Azumaya algebra. Therefore, $\mathsf{M}_c$ can be regarded as the moduli space of irreducible representations of $\mathsf{H}_{0,c}$.

Let $E \in \mathsf{M}_c$ be an irreducible representation of $\mathsf{H}_{0,c}$. We have seen before that $E$ has dimension $n!$ and is isomorphic to the regular representation as a representation of $\mathfrak{S}_n$. Let $\mathfrak{S}_{n-1} \subset \mathfrak{S}_n$ be the subgroup which preserves the element 1. Then the space of invariants $E^{\mathfrak{S}_{n-1}}$ has dimension $n$. On this space we have operators $X, Y : E^{\mathfrak{S}_{n-1}} \to E^{\mathfrak{S}_{n-1}}$ obtained by restriction of the operators $x_1, y_1$ on $E$ to the subspace of invariants. We have

$$[X, Y] = T := \sum_{i=2}^{n} s_{1i}.$$

Let us now calculate the right hand side of this equation explicitly. Let $\mathsf{e}$ be the symmetrizer of $\mathfrak{S}_{n-1}$. Let us realize the regular representation $E$ of $\mathfrak{S}_n$ as $\mathbb{C}[\mathfrak{S}_n]$ with action of $\mathfrak{S}_n$ by left multiplication. Then $v_1 = \mathsf{e}, v_2 = \mathsf{e}s_{12}, \ldots, v_n = \mathsf{e}s_{1n}$ is a basis of $E^{\mathfrak{S}_{n-1}}$. The element $T$ commutes with $\mathsf{e}$, so we have

$$Tv_i = \sum_{j \neq i} v_j.$$

This means that $T+1$ has rank 1, and hence the pair $(X, Y)$ defines a point on the Calogero-Moser space $\mathcal{C}_n$. [8]

[8]Note that the pair $(X, Y)$ is well defined only up to conjugation, because the representation $E$ is well defined only up to an isomorphism.

We now set $(X,Y)=f(E)$. It is clear that $f:\mathsf{M}_c\to\mathcal{C}_n$ is a regular map. So it remains to show that $f$ is an isomorphism. This is equivalent to showing that the corresponding map of function algebras $f^*:\mathcal{O}(\mathcal{C}_n)\to\mathsf{B}_{0,c}$ is an isomorphism.

Let us calculate $f$ and $f^*$ more explicitly. To do so, consider the open set $\mathsf{U}$ in $\mathsf{M}_c$ consisting of representations in which $x_i-x_j$ acts invertibly. These are exactly the representations that are obtained by restricting representations of $\mathfrak{S}_n\ltimes\mathbb{C}[x_1,\dots,x_n,p_1,\dots,p_n,\delta(\mathbf{x})^{-1}]$ using the classical Dunkl embedding. Thus representations $E\in\mathsf{U}$ are of the form $E=E_{\lambda,\mu}$ ($\lambda,\mu\in\mathbb{C}^n$, and $\lambda$ has distinct coordinates), where $E_{\lambda,\mu}$ is the space of complex valued functions on the orbit $\mathcal{O}_{\lambda,\mu}\subset\mathbb{C}^{2n}$, with the following action of $\mathsf{H}_{0,c}$:

$$(x_iF)(\mathbf{a},\mathbf{b})=a_iF(\mathbf{a},\mathbf{b}),\quad (y_iF)(\mathbf{a},\mathbf{b})=b_iF(\mathbf{a},\mathbf{b})+\sum_{j\neq i}\frac{(s_{ij}F)(\mathbf{a},\mathbf{b})}{a_i-a_j}.$$

(the group $\mathfrak{S}_n$ acts by permutations).

Now let us consider the space $E_{\lambda,\mu}^{\mathfrak{S}_{n-1}}$. A basis of this space is formed by characteristic functions of $\mathfrak{S}_{n-1}$-orbits on $\mathcal{O}_{\lambda,\mu}$. Using the above presentation, it is straightforward to calculate the matrices of the operators $X$ and $Y$ in this basis:

$$X=\mathrm{diag}(\lambda_1,\dots,\lambda_n),$$

and

$$Y_{ij}=\mu_i \text{ if } j=i,\ Y_{ij}=\frac{1}{\lambda_i-\lambda_j} \text{ if } j\neq i.$$

This shows that $f$ induces an isomorphism $f|_{\mathsf{U}}:\mathsf{U}\to U_n$, where $U_n$ is the subset of $\mathcal{C}_n$ consisting of pairs $(X,Y)$ for which $X$ has distinct eigenvalues.

From this presentation, it is straigtforward that $f^*(\mathrm{Tr}\,(X^p))=x_1^p+\cdots+x_n^p$ for every positive integer $p$. Also, $f$ commutes with the natural $\mathrm{SL}_2(\mathbb{C})$-action on $\mathsf{M}_c$ and $\mathcal{C}_n$ (by $(X,Y)\to(aX+bY,cX+dY)$), so we also get $f^*(\mathrm{Tr}\,(Y^p))=y_1^p+\cdots+y_n^p$, and

$$f^*(\mathrm{Tr}\,(X^pY))=\frac{1}{p+1}\sum_{m=0}^{p}\sum_i x_i^m y_i x_i^{p-m}.$$

Now, using the necklace bracket formula on $\mathcal{C}_n$ and the commutation relations of $\mathsf{H}_{0,c}$, we find, by a direct computation, that $f^*$ preserves Poisson bracket on the elements $\mathrm{Tr}\,(X^p)$, $\mathrm{Tr}\,(X^qY)$. But these elements are a local coordinate system near a generic point, so it follows that $f$ is a Poisson map. Since the algebra $\mathsf{B}_{0,c}$ is Poisson generated by $\sum x_i^p$ and $\sum y_i^p$ for all $p$, we get that $f^*$ is a surjective map.

Also, $f^*$ is injective. Indeed, by Wilson's theorem the Calogero-Moser space is connected, and hence the algebra $\mathcal{O}(\mathcal{C}_n)$ has no zero divisors, while $\mathcal{C}_n$ has the same dimension as $\mathsf{M}_c$. This proves that $f^*$ is an isomorphism, so $f$ is an isomorphism. □

9.7. **The Hilbert scheme $\mathrm{Hilb}_n(\mathbb{C}^2)$ and the Calogero-Moser space.** The Hilbert scheme $\mathrm{Hilb}_n(\mathbb{C}^2)$ is defined to be

$$\begin{aligned}&\mathrm{Hilb}_n(\mathbb{C}^2)=\{\text{ ideals } I\subset\mathbb{C}[x,y] | \mathrm{codim} I=n\}\\ =\ &\{(E,v)|E \text{ is a } \mathbb{C}[x,y]\text{-module of dimension } n,\ v \text{ is a cyclic vector of } E\}.\end{aligned}$$

The second equality can be easily seen from the short exact sequence

$$0\to I\to\mathbb{C}[x,y]\to E\to 0.$$

Let $S^{(n)}\mathbb{C}^2 = \underbrace{\mathbb{C}^2 \times \cdots \times \mathbb{C}^2}_{n \text{ times}}/\mathfrak{S}_n$, where $\mathfrak{S}_n$ acts by permutation. We have a natural map $\mathrm{Hilb}_n(\mathbb{C}^2) \to S^{(n)}\mathbb{C}^2$ which sends every ideal $I$ to its zero set (with multiplicities). This map is called the *Hilbert-Chow map*.

**Theorem 9.17** (Fogarty, [F])**.** (i) $\mathrm{Hilb}_n(\mathbb{C}^2)$ *is a smooth quasiprojective variety.*
(ii) *The Hilbert-Chow map* $\mathrm{Hilb}_n(\mathbb{C}^2) \to S^{(n)}\mathbb{C}^2$ *is projective. It is a resolution of singularities.*

*Proof.* Proof can be found in [Na]. □

Now consider the Calogero-Moser space $\mathcal{C}_n$ defined in Section 9.2.

**Theorem 9.18** (see [Na])**.** *The Hilbert Scheme* $\mathrm{Hilb}_n(\mathbb{C}^2)$ *is* $C^\infty$*-diffeomorphic to* $\mathcal{C}_n$*.*

**Remark 9.19.** More precisely there exists a family of algebraic varieties over $\mathbb{A}_1$, say $X_t$, $t \in \mathbb{A}_1$, such that $X_t$ is isomorphic to $\mathcal{C}_n$ if $t \neq 0$ and $X_0$ is the Hilbert scheme; and also if we denote by $\overline{X_t}$ the deformation of $\mathbb{C}^{2n}/\mathfrak{S}_n$ into the Calogero-Moser space, then there exists a map $f_t : X_t \twoheadrightarrow \overline{X_t}$, such that for $t \neq 0$, $f_t$ is an isomorphism and $f_0$ is the Hilbert-Chow map.

**Remark 9.20.** Consider the action of $G = \mathrm{PGL}_n$ on $T^*\mathrm{M}at_n$. As we have discussed, the corresponding moment map is $\mu(X,Y) = [X,Y]$, so $\mu^{-1}(0) = \{(X,Y)|[X,Y] = 0\}$ is the commuting variety. We can consider two kinds of quotient $\mu^{-1}(0)/G$ (i.e., of Hamiltonian reduction):

(1) The categorical quotient, i.e.,

$$\mathrm{Spec}(\mathbb{C}[x_{ij}, y_{ij}]/\langle[X,Y] = 0\rangle)^G \cong (\mathbb{C}^n \times \mathbb{C}^n)/\mathfrak{S}_n.$$

It is a reduced (by Gan-Ginzburg Theorem 9.14), affine but singular variety.

(2) The GIT quotient, in which the stability condition is that there exists a cyclic vector for $X, Y$. This quotient is $\mathrm{Hilb}_n(\mathbb{C}^2)$, which is smooth but not affine.

Both of these reductions are degenerations of the reduction along the orbit of matrices $T$ such that $T+1$ has rank 1, which yields the space $\mathcal{C}_n$. This explains why Theorem 9.18 and the results mentioned in Remark 9.19 are natural to expect.

9.8. **The cohomology of $\mathcal{C}_n$.** We also have the following result describing the cohomology of $\mathcal{C}_n$ (and hence, by Theorem 9.18, of $\mathrm{Hilb}_n(\mathbb{C}^2)$). Define the *age filtration* for the symmetric group $\mathfrak{S}_n$ by setting

$$\mathrm{age(transposition)} = 1.$$

Then one can show that for any $\sigma \in \mathfrak{S}_n$, $\mathrm{age}(\sigma) = \mathrm{rank}\,(1-\sigma)|_{\text{reflection representation}}$. It is easy to see that $0 \leq \mathrm{age} \leq n-1$. Notice also that the age filtration can be defined for any Coxeter group.

**Theorem 9.21** (Lehn-Sorger, Vasserot)**.** *The cohomology ring* $\mathrm{H}^*(\mathcal{C}_n, \mathbb{C})$ *lives in even degrees only and is isomorphic to* $\mathrm{gr}(\mathrm{C}enter(\mathbb{C}[\mathfrak{S}_n]))$ *under the age filtration (with doubled degrees).*

*Proof.* Let us sketch a noncommutative-algebraic proof of this theorem, given in [EG]. This proof is based on the following result.

**Theorem 9.22** (Nest-Tsygan, [NT]). *If $M$ is an affine symplectic variety and $A$ is a quantization of $M$, then*

$$\mathrm{HH}^*(A[\hbar^{-1}], A[\hbar^{-1}]) \cong \mathrm{H}^*(M, \mathbb{C}((\hbar)))$$

*as an algebra over $\mathbb{C}((\hbar))$.*

Now, we know that the algebra $\mathsf{B}_{t,c}$ is a quantization of $\mathcal{C}_n$. Therefore by above theorem, the cohomology algebra of $\mathcal{C}_n$ is the cohomology of $\mathsf{B}_{t,c}$ (for generic $t$). But $\mathsf{B}_{t,c}$ is Morita equivalent to $\mathsf{H}_{t,c}$, so this cohomology is the same as the Hochschild cohomology of $\mathsf{H}_{t,c}$. However, the latter can be computed by using that $\mathsf{H}_{t,c}$ is given by generators and relations (by producing explicit representatives of cohomology classes and computing their product), which gives the result. □

9.9. **Notes.** Sections 9.1–9.6 follow Section 1, 2, 4 of [E4]; the parts about the Hilbert scheme and its relation to Calogero-Moser spaces follow the book [Na] (see also [GS]); the other parts follow the paper [EG].

18.735 Double Affine Hecke Algebras in Representation Theory, Combinatorics, Geometry, and Mathematical Physics
Fall 2009