

6. THE KNIZHNIK-ZAMOŁODCHIKOV FUNCTOR

6.1. Braid groups and Hecke algebras. Let G be a complex reflection group and let \mathfrak{h} be its reflection representation. For any reflection hyperplane $H \subset \mathfrak{h}$, its pointwise stabilizer is a cyclic group of order m_H . Fix a collection of nonzero constants $q_{1,H}, \dots, q_{m_H-1,H}$ which are G -invariant, namely, if H and H' are conjugate to each other under some element in G , then $q_{i,H} = q_{i,H'}$ for $i = 1, \dots, m_H - 1$.

Let $B_G = \pi_1(\mathfrak{h}_{\text{reg}}/G, \mathbf{x}_0)$ be the braid group of G , and $T_H \in B_G$ be a representative of the conjugacy class defined by a small circle around the image of H in \mathfrak{h}/G oriented in the counterclockwise direction.

The following theorem follows from elementary algebraic topology.

Proposition 6.1. *The group G is the quotient of the braid group B_G by the relations*

$$T_H^{m_H} = 1$$

for all reflection hyperplanes H .

Proof. See, e.g., [BMR] Proposition 2.17. □

Definition 6.2. The Hecke algebra of G is defined to be

$$\mathcal{H}_q(G) = \mathbb{C}[B_G] / \langle (T_H - 1) \prod_{j=1}^{m_H-1} (T_H - \exp(2\pi i j/m_H) q_{j,H}), \text{ for all } H \rangle.$$

Thus, by Proposition 6.1 we have an isomorphism

$$\mathcal{H}_1(G) \cong \mathbb{C}G.$$

So $\mathcal{H}_q(G)$ is a deformation of $\mathbb{C}G$.

Example 6.3 (Coxeter group case). Now let W be a Coxeter group. Let \mathcal{S} be the set of reflections and let $\alpha_s = 0$ be the reflection hyperplane corresponding to $s \in \mathcal{S}$. The Hecke algebra $\mathcal{H}_q(W)$ is the quotient of $\mathbb{C}[B_W]$ by the relations

$$(T_s - 1)(T_s + q_s) = 0,$$

for all reflections s where T_s is a small counterclockwise circle around the image of the hyperplane $\alpha_s = 0$ in \mathfrak{h}/W .

6.2. KZ functors. For a complex reflection group G , let $\text{Loc}(\mathfrak{h}_{\text{reg}})$ be the category of local systems (i.e., \mathcal{O} -coherent \mathcal{D} -modules) on $\mathfrak{h}_{\text{reg}}$, and let $\text{Loc}(\mathfrak{h}_{\text{reg}})^G$ be the category of G -equivariant local systems on $\mathfrak{h}_{\text{reg}}$, i.e. of local systems on $\mathfrak{h}_{\text{reg}}/G$.

Suppose that $G' = 1$ is the trivial subgroup in G . Then the restriction functor defined in Section 5.10 defines a functor $\text{Res} : \mathcal{O}_c(G, \mathfrak{h})_0 \rightarrow \text{Loc}(\mathfrak{h}_{\text{reg}}/G)$. Also, we have the monodromy functor $\text{Mon} : \text{Loc}(\mathfrak{h}_{\text{reg}}/G) \cong \text{Rep}(B_G)$. The composition of these two functors is a functor from $\mathcal{O}_c(G, \mathfrak{h})_0$ to $\text{Rep}(B_G)$, which is exactly the KZ functor defined in [GGOR]. We will denote this functor by KZ.

Theorem 6.4 (Ginzburg, Guay, Opdam, Rouquier, [GGOR]). *The KZ functor factors through*

$\text{Rep}\mathcal{H}_q(G)$, where

$$q_{j,H} = \exp(2\pi i b_{j,H}/m_H), \quad \text{and} \quad b_{j,H} = 2 \sum_{\ell=1}^{m_H-1} \frac{c_{s_H^\ell} (1 - e^{2\pi i j \ell/m_H})}{1 - e^{-2\pi i \ell/m_H}}.$$

Proof. Assume first that c is generic. Then the category $\mathcal{O}_c(G, \mathfrak{h})_0$ is semisimple, with simple objects $M_c(\tau)$, so it is enough to check the statement on $M_c(\tau)$. Consider the trivial bundle over $\mathfrak{h}_{\text{reg}}$ with fiber τ . The KZ connection on it has the form

$$d - \sum_{s \in \mathcal{S}} \frac{2c_s}{1 - \lambda_s} \frac{d\alpha_s}{\alpha_s} (1 - s).$$

The residue of the connection form of this connection on the hyperplane H on the j -th irreducible representation of $\mathbb{Z}/m_H\mathbb{Z}$ is

$$2 \sum_{\ell=1}^{m_H-1} \frac{c_{s_H}^\ell}{1 - e^{-2\pi i \ell / m_H}} (1 - e^{2\pi i j \ell / m_H}).$$

Therefore, the monodromy operator around this hyperplane is diagonalizable, and the eigenvalues of this operator are 1 and $\exp(2\pi i j / m_H) q_{j,H}$, as desired.

For special c , introduce the generalized Verma module

$$M_{c,n}(\tau) = H_c(G, \mathfrak{h}) \otimes_{\mathbb{C}G \ltimes S\mathfrak{h}} (\tau \otimes S\mathfrak{h}/\mathfrak{m}^{n+1}),$$

where $\mathfrak{m} \subset S\mathfrak{h}$ is the maximal ideal of 0, $n \geq 0$. Clearly, $M_{c,0} = M_c(\tau)$. Moreover, $M_{c,n} \in \mathcal{O}_c(G, \mathfrak{h})_0$ for any n , since it has a finite filtration whose successive quotients are Verma modules.

Theorem 6.5. *For large enough n , $M_{c,n}(\mathbb{C}G)$ contains a direct summand which is a projective generator of $\mathcal{O}_c(G, \mathfrak{h})_0$.*

Proof. From the definition, $M_{c,n} = S\mathfrak{h}^* \otimes \mathbb{C}G \otimes S\mathfrak{h}/\mathfrak{m}^{n+1}$. Let ∂ be the degree operator on $M_{c,n}(\mathbb{C}G)$ with $\deg \mathfrak{h}^* = 1$, $\deg \mathfrak{h} = -1$, and $\deg G = 0$, i.e., we have

$$[\partial, x] = x, [\partial, y] = -y, \text{ where } x \in \mathfrak{h}^*, y \in \mathfrak{h}.$$

So $\mathfrak{h} - \partial$ is a module endomorphism of $M_{c,n}(\mathbb{C}G)$ where \mathfrak{h} is the operator defined in (3.2). Moreover, $\mathfrak{h} - \partial$ acts locally finitely. In particular, we have a decomposition of $M_{c,n}(\mathbb{C}G)$ into generalized eigenspaces of $\mathfrak{h} - \partial$:

$$M_{c,n}(\mathbb{C}G) = \bigoplus_{\beta \in \mathbb{C}} M_{c,n}^\beta(\mathbb{C}G).$$

We have

$$\text{Hom}(M_{c,n}(\mathbb{C}G), N) = \{\text{vectors in } N \text{ which are killed by } \mathfrak{m}^{n+1}\},$$

and

$$\text{Hom}(M_{c,n}^\beta(\mathbb{C}G), N) = \{\text{vectors in } N \text{ which are killed by } \mathfrak{m}^{n+1}$$

and are generalized eigenvectors of \mathfrak{h} with generalized eigenvalue $\beta\}$.

Let $\Sigma = \{h_c(\tau) | \tau \text{ is a irreducible representation of } G\}$ (recall that $h_c(\tau) = \frac{\dim \mathfrak{h}}{2} - \sum_{s \in \mathcal{S}} \frac{2c_s}{1 - \lambda_s} s |_\tau$), and let

$$M_{c,n}^\Sigma(\mathbb{C}G) = \bigoplus_{\beta \in \Sigma} M_{c,n}^\beta(\mathbb{C}G).$$

Claim: for large n , $M_{c,n}^\Sigma(\mathbb{C}G)$ is a projective generator of $\mathcal{O}_c(G, \mathfrak{h})_0$.

Proof of the claim. First, for any β , there exists n such that $M_{c,n}^\Sigma(\mathbb{C}G)$ is projective (since the condition of being killed by \mathfrak{m}^{n+1} is vacuous for large n).

Secondly, consider the functor $\text{Hom}(M_{c,n}^\Sigma(\mathbb{C}G), \bullet)$. For any module $N \in \mathcal{O}_c(G, \mathfrak{h})_0$, if $\text{Hom}(M_{c,n}^\Sigma(\mathbb{C}G), N) = 0$, then $\bigoplus_{\beta \in \Sigma} N[\beta] = 0$. So $N = 0$. Thus this functor does not kill nonzero objects, and so $M_{c,n}^\Sigma(\mathbb{C}G)$ is a generator. \square

Theorem 6.5 is proved. \square

Corollary 6.6. (i) $\mathcal{O}_c(G, \mathfrak{h})_0$ has enough projectives, so it is equivalent to the category of modules over a finite dimensional algebra.

(ii) Any object of $\mathcal{O}_c(G, \mathfrak{h})_0$ is a quotient of a multiple of $M_{c,n}(\mathbb{C}G)$ for large enough n .

Proof. Directly from the definition and the above theorem. \square

Now we can finish the proof of Theorem 6.4. We have shown that for generic c , $\text{KZ}(M_{c,n}(\mathbb{C}G)) \in \text{Rep}\mathcal{H}_q(G)$. Hence this is true for any c , since $M_{c,n}(\mathbb{C}G)$ is a flat family of modules over $H_c(G, \mathfrak{h})$. Then, $\text{KZ}(M)$ is a $\mathcal{H}_q(G)$ -module for all M , since any M is a quotient of $M_{c,n}(\mathbb{C}G)$ and the functor KZ is exact. \square

Corollary 6.7 (Broué, Malle, Rouquier, [BMR]). Let $q_{j,H} = \exp(t_{j,H})$ where $t_{j,H}$'s are formal parameters. Then $\mathcal{H}_q(G)$ is a free module over $\mathbb{C}[[t_{j,H}]]$ of rank $|G|$.

Proof. We have

$$\mathcal{H}_q(G)/(t) = \mathcal{H}_1(G) = \mathbb{C}G.$$

So it remains to show that $\mathcal{H}_q(G)$ is free. To show this, it is sufficient to show that any $\tau \in \text{Irrep}G$ admits a flat deformation τ_q to a representation of $\mathcal{H}_q(G)$. We can define this deformation by letting $\tau_q = \text{KZ}(M_c(\tau))$. \square

Remark 6.8. 1. The validity of this Corollary in characteristic zero implies that it is also valid over a field positive characteristic.

2. It is not known in general if the Corollary holds for numerical q (even generically). This is a conjecture of Broué, Malle, and Rouquier. But it is known for many cases (including all Coxeter groups).

3. The proof of the Corollary is analytic (it is based on the notion of monodromy). There is no known algebraic proof, except in special cases, and in the case of Coxeter groups, which we'll discuss later.

6.3. The image of the KZ functor. First, let us recall the definition of a quotient category. Let \mathcal{A} be an abelian category and $\mathcal{B} \subset \mathcal{A}$ a full abelian subcategory.

Definition 6.9. \mathcal{B} is a Serre subcategory if it is closed under subquotients and extensions (i.e., if two terms in a short exact sequence are in \mathcal{B} , so is the third one).

If $\mathcal{B} \subset \mathcal{A}$ is a Serre subcategory, one can define a category \mathcal{A}/\mathcal{B} as follows:

$$\begin{aligned} \text{objects in } \mathcal{A}/\mathcal{B} &= \text{objects in } \mathcal{A}, \\ \text{Hom}_{\mathcal{A}/\mathcal{B}}(X, Y) &= \varinjlim_{Y', X/X' \in \mathcal{B}} \text{Hom}_{\mathcal{A}}(X', Y/Y'). \end{aligned}$$

The category \mathcal{A}/\mathcal{B} is an abelian category with the following universal property: any exact functor $F : \mathcal{A} \rightarrow \mathcal{C}$ that kills \mathcal{B} must factor through \mathcal{A}/\mathcal{B} .

Now let $\mathcal{O}_c(G, \mathfrak{h})_0^{\text{tor}}$ be the full subcategory of $\mathcal{O}_c(G, \mathfrak{h})_0$ consisting of modules supported on the reflection hyperplanes. It is a Serre subcategory, and $\ker(\text{KZ}) = \mathcal{O}_c(G, \mathfrak{h})_0^{\text{tor}}$. Thus we have a functor:

$$\overline{\text{KZ}} : \mathcal{O}_c(G, \mathfrak{h})_0 / \mathcal{O}_c(G, \mathfrak{h})_0^{\text{tor}} \rightarrow \text{Rep}\mathcal{H}_q(G).$$

Theorem 6.10 (Ginzburg, Guay, Opdam, Rouquier, [GGOR]). *If $\dim \mathcal{H}_q(G) = |G|$, the functor $\overline{\text{KZ}}$ is an equivalence of categories.*

Proof. See [GGOR], Theorem 5.14. □

6.4. Example: the symmetric group \mathfrak{S}_n . Let $\mathfrak{h} = \mathbb{C}^n$, $G = \mathfrak{S}_n$. Then we have (for $q \in \mathbb{C}^*$):

$$\mathcal{H}_q(\mathfrak{S}_n) = \langle T_1, \dots, T_{n-1} \rangle / \langle \text{the braid relations and } (T_i - 1)(T_i + q) = 0 \rangle.$$

The following facts are known:

- (1) $\dim \mathcal{H}_q(\mathfrak{S}_n) = n!$;
- (2) $\mathcal{H}_q(\mathfrak{S}_n)$ is semisimple if and only if $\text{ord}(q) \neq 2, 3, \dots, n$.

Now suppose q is generic. Let λ be a partition of n . Then we can define an $\mathcal{H}_q(\mathfrak{S}_n)$ -module S_λ , the Specht module for the Hecke algebra in the sense of [DJ]. This is a certain deformation of the classical irreducible Specht module for the symmetric group. The Specht module carries an inner product $\langle \cdot, \cdot \rangle$. Denote $D_\lambda = S_\lambda / \text{Rad}\langle \cdot, \cdot \rangle$.

Theorem 6.11 (Dipper, James, [DJ]). *D_λ is either an irreducible $\mathcal{H}_q(\mathfrak{S}_n)$ -module, or 0. $D_\lambda \neq 0$ if and only if λ is e -regular where $e = \text{ord}(q)$ (i.e., every part of λ occurs less than e times).*

Proof. See [DJ], Theorem 6.3, 6.8. □

Now let $M_c(\lambda)$ be the Verma module associated to the Specht module for \mathfrak{S}_n and $L_c(\lambda)$ be its irreducible quotient. Then we have the following theorem.

Theorem 6.12. *If $c \leq 0$, then $\text{KZ}(M_c(\lambda)) = S_\lambda$ and $\text{KZ}(L_c(\lambda)) = D_\lambda$.*

Proof. See Section 6.2 of [GGOR]. □

Corollary 6.13. *If $c \leq 0$, then $\text{Supp}L_c(\lambda) = \mathbb{C}^n$ if and only if λ is e -regular. If $c > 0$, then $\text{Supp}L_c(\lambda) = \mathbb{C}^n$ if and only if λ^\vee is e -regular, or equivalently, λ is e -restricted (i.e., $\lambda_i - \lambda_{i+1} < e$ for $i = 1, \dots, n-1$).*

Proof. Directly from the definition and the above theorem. □

6.5. Notes. The references for this section are [GGOR], [BMR].

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