GRASSMANNIANS: THE FIRST EXAMPLE OF A MODULI SPACE

1. What is this course about?

Many objects of algebraic geometry, such as subspaces of a linear space, smooth curves of genus g, or stable vector bundles on a curve, themselves vary in algebraically defined families. Moduli theory studies such families of algebraic objects.

Roughly speaking a *moduli problem* is the problem of understanding a given geometrically meaningful functor from the category of schemes to sets. To make this more concrete consider the following three functors.

Example 1.1 (Example 1: The Grassmannian Functor.)**.** Let S be a scheme, E a vector bundle on S and k a positive integer less than the rank of E. Let

$$Gr(k, S, E) : \{\text{Schemes}/S\} \to \{\text{sets}\}$$

be the contravariant functor that associates to an S-scheme X subvector bundles of rank k of $X \times_S E$.

Example 1.2 (Example 2: The Hilbert Functor.)**.** Let $X \to S$ be a projective scheme, $\mathcal{O}(1)$ a relatively ample line bundle and P a fixed polynomial. Let

$$Hilb_P(X/S) : \{\text{Schemes}/S\} \to \{\text{sets}\}$$

be the contravariant functor that associates to an S scheme Y the subschemes of $X \times_S Y$ which are proper and flat over Y and have the Hilbert polynomial P.

Example 1.3 (Example 3: Moduli of stable curves.)**.** Let

$$\overline{\mathcal{M}}_g : \{\text{Schemes}\} \to \{\text{sets}\}$$

be the functor that assigns to a scheme Z the set of families (up to isomorphism) $X \to Z$ flat over Z whose fibers are stable curves of genus g.

Each of the functors in the three examples above poses a moduli problem. The first step in the solution of such a problem is to construct a smooth, projective variety/proper scheme/ proper Deligne-Mumford stack that represents the functor finely/coarsely.

Definition 1.4. Given a contravariant functor F from schemes over S to sets, we say that a scheme $X(F)$ over S and an element $U(F) \in F(X(F))$ represents the functor finely if for every S scheme Y the map

$$\text{Hom}_S(Y, X(F)) \to F(Y)$$

given by $g \to g^*U(F)$ is an isomorphism.

The best answer one can usually hope for (such as in Examples 1 and 2) is that there is a scheme (hopefully proper) representing the functor. There may not be such a scheme. For instance for the functor in Example 3 there does not exist a fine moduli scheme representing the functor. In such cases we represent the functor either in a different category or we relax the conditions that we impose on the representing scheme. The most common alternatives are to work with stacks or to ask for the moduli space to only coarsely represent the functor.

Definition 1.5. Given a contravariant functor F from schemes over S to sets, we say that a scheme $X(F)$ over S coarsely represents the functor F if there is a natural transformation of functors $\Phi : F \to \mathrm{Hom}_S(*, X(F))$ such that

(1) $\Phi(spec(k)) : F(spec(k)) \to \mathrm{Hom}_S(spec(k), X(F))$ is a bijection for every algebraically closed field k,
(2) For any S-scheme Y and any natural transformation $\Psi : F \to \mathrm{Hom}_S(*, Y)$, there is a unique natural transformation

$$\Pi : \mathrm{Hom}_S(*, X(F)) \to \mathrm{Hom}_S(*, Y)$$

such that $\Psi = \Pi \circ \Phi$.

Finding a representing scheme/stack, a moduli space, is only the first step of a moduli problem. Usually the motivation for constructing a moduli space is to understand the objects this space parameterizes. This in turn requires a good knowledge of the geometry of the moduli space. Among the questions that arise about these moduli spaces are:

(1) Is the moduli space proper? If not, does it have a modular compactification? Is the moduli space projective?
(2) What is the dimension of the moduli space? Is it connected? Is it irreducible? What are its singularities?
(3) What is the cohomology/Chow ring of the moduli space?
(4) What is the Picard group of the moduli space? Assuming the moduli space is projective, which of the divisors are ample? Which of the divisors are effective?
(5) Can the moduli space be rationally parameterized? What is the Kodaira dimension of the moduli space?

The second step of the moduli problem is answering as many of these questions as possible. The focus of this course will be the second step of the moduli problem. In this course we will not concentrate on the constructions of the moduli spaces. We will often stop at outlining the main steps of the constructions only in so far as they help us understand the geometry. We will spend most of the time talking about the explicit geometry of these moduli spaces.

We begin our study with the Grassmannian. The Grassmannian is the scheme that represents the functor in Example 1. Grassmannians lie at the heart of moduli theory. Their existence is the first step for the proof of the existence of the Hilbert scheme. Many moduli spaces in turn can be constructed using the Hilbert scheme. On the other hand, the Grassmannians are sufficiently simple that their geometry is well-understood. Many of the constructions for understanding the geometry of other moduli spaces, such as the moduli space of stable curves, imitates the techniques used in the case of Grassmannians. This motivates us to begin our exploration with the Grassmannian.

Additional references: For a more detailed introduction to moduli problems you might want to read [HM] Chapter 1 Section A, [H] Lecture 21, [EH] Section VI and [K] Section I.1.

2. Preliminaries about the Grassmannian

Good references for this section (in random order) are [H] Lectures 6 and 16, [GH] I.5 and [Ful2] Chapter 14, [Kl2] and [KL].

Let $G(k,n)$ denote the classical Grassmannian that parameterizes k-dimensional linear subspaces of a fixed n-dimensional vector space V. $G(k,n)$ naturally carries the structure of a smooth, projective variety. It is often convenient to think of $G(k,n)$ as the parameter space of $k-1$-dimensional projective linear spaces in $\mathbb{P}^{n-1}$. When we use this point of view, we will denote the Grassmannian by $\mathbb{G}(k-1,n-1)$.

It is easy to give $G(k,n)$ the structure of an abstract variety. In case $V=\mathbb{C}^n$, $G(k,n)$ becomes a complex manifold under this structure. Given a k-dimensional subspace Ω of V, we can represent it by a $k \times n$ matrix. Choose a basis for Ω and write them as the row vectors of the matrix. $GL(k)$ acts on the left by multiplication. Two $k \times n$ matrices represent the same linear space if and only if they are related by this action of $GL(k)$. Since the k vectors span Ω, in the matrix representation there must exist a non-vanishing $k \times k$ minor. Suppose we look at those matrices that have a fixed non-vanishing $k \times k$ minor. We can normalize that this submatrix is the identity matrix. This gives a unique representation for Ω. In this representation the remaining entries are free to vary. The space of such matrices is isomorphic to $\mathbb{A}^{k(n-k)}$. In case $V=\mathbb{C}^n$ the transition functions are clearly holomorphic. We thus obtain the structure of a complex manifold of dimension $k(n-k)$ on the Grassmannian $G(k,n)$. The Grassmannian is compact and connected (for example, the unitary group $U(n)$ maps continuously onto $G(k,n)$).

The cohomology/Chow ring of the Grassmannian can be very explicitly described. Fix a flag

$$F_\bullet : 0 = F_0 \subset F_1 \subset \cdots \subset F_n = V$$

in the vector space V. Recall that a flag is a nested sequence of vector subspaces of V where the difference in dimension of two consecutive vector spaces is one. Given a partition λ with k parts satisfying the conditions

$$n-k \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0,$$

we can define a subvariety of the Grassmannian called the Schubert variety $\Sigma_{\lambda_1,\ldots,\lambda_k}(F_\bullet)$ of type λ with respect to the flag $F_\bullet$ to be the closure of

$$\Sigma^0_{\lambda_1,\ldots,\lambda_k}(F_\bullet) := \{\ [\Omega] \in G(k,n)\ :\ \dim(\Omega \cap F_{n-k+i-\lambda_i}) = i\ \}.$$

This closure is obtained by turning all $=$ to $\geq$ in the rank conditions.

The homology class of a Schubert variety does not depend on the choice of flag. For each partition satisfying the above properties, we get a homology class. A word of caution: Most Schubert varieties in a Grassmannian are singular.

Fix the standard flag for $\mathbb{C}^n$ where $F_i =< e_1,\ldots,e_i >$. Let Ω be a k-plane in the open part of the Schubert variety $\Sigma_{\lambda_1,\ldots,\lambda_k}(F_\bullet)$ defined with respect to this flag. We can normalize $\Omega \cap F_{n-k+i-\lambda_i}$ so that $< v_i, e_{n-k+j-\lambda_j} >= 0$ for $j<i$ and $< v_i, e_{n-k+i-\lambda_i} >= 1$. Thus we get a unique matrix representation for Ω and see that $\Sigma^0_{\lambda_1,\dots,\lambda_k}(F_\bullet) \cong \mathbb{A}^{k(n-k)-\sum_i \lambda_i}$. In other words, Schubert varieties give a cell-decomposition of $G(k,n)$ with only even dimensional cells. It follows that the classes of Schubert varieties generates the homology of $G(k,n)$. Applying Poincaré duality we obtain the following fundamental theorem about the cohomology of $G(k,n)$. We will denote the cohomology class that corresponds to the Schubert variety $\Sigma_{\lambda_1,\dots,\lambda_k}$ by $\sigma_{\lambda_1,\dots,\lambda_k}$. We often omit the indices that are zero.

Theorem 2.1. *The Poincaré duals of the classes of Schubert varieties give an additive basis of the cohomology of the Grassmannian.*

Example 2.2. Let us consider the case $G(2,4) = \mathbb{G}(1,3)$. This variety geometrically corresponds to the variety of lines in $\mathbb{P}^3$. The Schubert varieties are in this case given by Σ_1 in codimension 1, $\Sigma_{1,1}$ and Σ_2 in codimension 2, $\Sigma_{2,1}$ in codimension 3 and $\Sigma_{2,2}$ in codimension 4. Of course, all the codimensions are complex codimensions. A flag in $\mathbb{P}^3$ corresponds to a choice of point q contained in a line l contained in a plane P contained in $\mathbb{P}^3$. Σ_1 parameterizes lines that intersect l. Σ_2 parameterizes lines that contain q. $\Sigma_{1,1}$ parameterizes lines that are contained in P. $\Sigma_{2,1}$ parameterizes lines that are contained in P and contain q.

Since the cohomology of Grassmannians is generated by Schubert cycles, given two Schubert cycles σ_λ and σ_μ, their product in the cohomology ring can be expressed as a linear combination of Schubert cycles.

$$\sigma_\lambda \cdot \sigma_\mu = \sum_\nu c^\nu_{\lambda,\mu}\ \sigma_\nu$$

The structure constants $c^\nu_{\lambda,\mu}$ of the cohomology ring with respect to the Schubert basis are known as Littlewood - Richardson coefficients.

Example 2.2 continued. Let us work out the Littlewood - Richardson coefficients of $G(2,4) = \mathbb{G}(1,3)$. All but one of the calculations are easy. It is simplest to work dually with the intersection of Schubert varieties. Suppose we wanted to calculate $\Sigma_2 \cap \Sigma_2$. Σ_2 is the class of lines that pass through a point. If we take points, there will be a unique line containing them both. We conclude that $\Sigma_2 \cap \Sigma_2 = \Sigma_{2,2}$. Similarly $\Sigma_{1,1} \cap \Sigma_{1,1} = \Sigma_{2,2}$, because there is a unique line contained in two distinct planes in $\mathbb{P}^3$. On the other hand $\Sigma_{1,1} \cap \Sigma_2 = 0$ since there will not be a line contained in a plane and passing through a point not contained in the plane.

The hardest class to compute is $\Sigma_1 \cap \Sigma_1$. We know that the class is expressible as a linear combination of $\Sigma_{1,1}$ and Σ_2. We just saw that both these cycles are self-dual. In order to compute the coefficient we can calculate the triple intersection. $\Sigma_1 \cap \Sigma_1 \cap \Sigma_2$ is the set of lines that meet two lines l_1, l_2 and contain a point q. There is a unique such line given by $\overline{ql_1} \cap \overline{ql_2}$. The other coefficient can be similarly computed to see $\sigma_1^2 = \sigma_{1,1} + \sigma_2$.

Exercise 2.3. Work out the multiplicative structure of the cohomology ring of $G(2,4) = \mathbb{G}(1,3)$, $G(2,5) = \mathbb{G}(1,4)$ and $G(3,6) = \mathbb{G}(2,5)$.

Exercise 2.4. Show that the dual of the Schubert cycle $\sigma_{\lambda_1,\dots,\lambda_k}$ is the Schubert cycle $\sigma_{n-k-\lambda_k,\dots,n-k-\lambda_1}$. Conclude that the Littlewood - Richardson coefficient $c^\nu_{\lambda,\mu}$ may be computed as the triple product $\sigma_\lambda \cdot \sigma_\mu \cdot \sigma_{\nu^*}$.

The method of undetermined coefficients we just employed is a powerful technique for calculating the classes of subvarieties of the Grassmannian. Let us do an example to show another use of the technique.

Example 2.5. How many lines are contained in the intersection of two general quadric hypersurfaces in $\mathbb{P}^4$? In order to work out this problem we can calculate the class of lines contained in a quadric hypersurface in $\mathbb{P}^4$ and square the class. The dimension of the space of lines on a quadric hypersurface is 3. The classes of dimension 3 in $\mathbb{G}(1,4)$ are given by σ_3 and $\sigma_{2,1}$. We can, therefore, write this class as $a\sigma_3 + b\sigma_{2,1}$. The coefficient of σ_3 is zero because σ_3 is self-dual and corresponds to lines that pass through a point. As long as the quadric hypersurface does not contain the point, the intersection will be zero. On the other hand, $b = 4$. $\Sigma_{2,1}$ parameterizes lines in $\mathbb{P}^4$ that intersect a $\mathbb{P}^1$ and are contained in a $\mathbb{P}^3$ containing the $\mathbb{P}^1$. The intersection of the quadric hypersurface with the $\mathbb{P}^3$ is a quadric surface. The lines have to be contained in this surface and must pass through the two points of intersection of the $\mathbb{P}^1$ with the quadric surface. There are four such lines. We conclude that there are 16 lines that are contained in the intersection of two general quadric hypersurfaces in $\mathbb{P}^4$.

Another way to verify this fact is to observe that such an intersection is a quartic Del Pezzo surface. Such a surface is the blow-up of $\mathbb{P}^2$ at 5 general points embedded by its anti-canonical linear system. The lines in this embedding correspond to the (-1)-curves on the surface. It is well-known that the number of (-1)-curves on this surface is 16 (see for example [Ha] Chapter 5).

There is one issue that requires some attention. So far we have pretended that all the intersections are transverse. This is indeed the case. We can either explicitly calculate the tangent spaces to check that the intersection is transverse or we can appeal to a general theorem that guarantees the result. Since the theorem is very useful, we reproduce its statement here. However, be warned that the theorem in the form stated holds only in characteristic zero. For a proof see [Kl1] or [Ha] Theorem III.10.8.

Theorem 2.6. *(Kleiman) Assume we are working over an algebraically closed field of characteristic zero. Let G be an integral algebraic group scheme, X an integral algebraic scheme with a transitive G action. Let $f: Y \to X$ and $g: Z \to X$ be two maps of integral algebraic schemes. For each rational element of $g \in G$, denote by gY the X-scheme given by $y \mapsto gf(y)$. Then there exists a dense open subset U of G such that for every rational element $g \in U$, the fiber product $(gY) \times_X Z$ is either empty or equidimensional of the expected dimension*

$$\dim(Y) + \dim(Z) - \dim(X).$$

Furthermore, if Y and Z are regular, for a dense open set this fibered product is regular.

Proof. The theorem follows from the following lemma.

Lemma 2.7. *Suppose all the schemes in the following diagram are integral over an algebraically closed field of characteristic zero.*

$$\begin{tikzcd} & W \arrow[dl, "p"'] \arrow[dr, "q"] & & Z \arrow[dl, "r"'] \\ S & & X & \end{tikzcd}$$

If q is flat, then there exists a dense open subset of S such that $p^{-1}(s) \times_X Z$ is empty or equidimensional of dimension

$$\dim(p^{-1}(s)) + \dim(Z) - \dim(X).$$

If in addition, Z is regular and q has regular fibers, then $p^{-1}(s) \times_X Z$ is regular for a dense open subset of S.

The theorem follows by taking $S = G$, $W = G \times Y$ and $q : G \times Y \to X$ given by $q(g, y) = gf(y)$. The lemma follows by flatness and generic smoothness. More precisely, since q is flat, the fibers of q are equidimensional of dimension $\dim(W) - \dim(X)$. By base change the induced map $W \times_X Z \to Z$ is also flat, hence the fibers have dimension $\dim(W \times_X Z) - \dim(Z)$. Consequently,

$$\dim(W \times_X Z) = \dim(W) + \dim(Z) - \dim(X).$$

There is an open subset $U_1 \subset S$ over which p is flat, so the fibers are either empty or equidimensional with dimension $\dim(W) - \dim(S)$. Similarly there is an open subset $U_2 \subset S$, where the fibers of $p \circ pr_W : X \times_X Z \to S$ is either empty or equidimensional of dimension $\dim(X \times_X Z) - \dim(S)$. The first part of the lemma follows by taking $U = U_1 \cap U_2$ and combining these dimension statements. The second statement follows by generic smoothness. This is where we use the assumption that the characteristic is zero. □

The Grassmannians $G(k, n)$ are homogeneous under the action of $GL(n)$. Hence Kleiman's Theorem easily implies the transversality of intersections in many cases.

We now give two presentations for the cohomology ring of the Grassmannian. These presentations are useful for theoretical computations. However, we will soon develop Littlewood - Richardson rules, positive combinatorial rules for computing Littlewood - Richardson coefficients, that are much more effective in computing and understanding the structure of the cohomology ring of $G(k, n)$.

One extremely useful way comes from considering the universal exact sequence of bundles on $G(k, n)$. Let T denote the tautological bundle over $G(k, n)$. Recall that the fiber of T over a point $[\Omega]$ is the vector subspace Ω of V. There is a natural inclusion

$$0 \to T \to \underline{V} \to Q \to 0$$

with quotient bundle Q.

Theorem 2.8. *As a ring the cohomology ring of $G(k, n)$ is isomorphic to*

$$\mathbb{R}[c_1(T), \dots, c_k(T), c_1(Q), \dots, c_{n-k}(Q)]/(c(T)c(Q) = 1).$$

Moreover, the chern classes of the Quotient bundle generate the cohomology ring.

The chern classes of the tautological bundle and the quotient bundle are easy to see in terms of Schubert cycles. As an exercise prove the following proposition:

Proposition 2.9. *The chern classes of the tautological bundle are given as follows:*

$$c_i(T) = (-1)^i \sigma_{1,\dots,1}$$

where there are i ones. The chern classes of the quotient bundle are given by

$$c_i(Q) = \sigma_i.$$

The Schubert cycles σ_i where all the parts of the partition except for the first are zero are called special Schubert cycles. It is easy to calculate the product of special Schubert cycles. Pieri's rule gives an algorithm for computing these products. In fact, Pieri's rule gives an algorithm for computing the product of any Schubert cycle with a special Schubert cycle.

Theorem 2.10 (Pieri's formula)**.** *Let σ_λ be a special Schubert cycle. Suppose σ_μ is any Schubert cycle with parts $\mu_1, \ldots, \mu_k$. Then*

$$\sigma_\lambda \cdot \sigma_\mu = \sum_{\substack{\mu_i \leq \nu_i \leq \mu_{i-1} \\ \sum \nu_i = \lambda + \sum \mu_i}} \sigma_\nu \tag{1}$$

The special Schubert cycles generate the cohomology ring of the Grassmannian. In order to prove this we have to express every Schubert cycle $\sigma_{\lambda_1,\ldots,\lambda_k}$ as a linear combination of products of special Schubert cycles.

Exercise 2.11. Using Pieri's formula prove the following identity

$$(-1)^k \sigma_{\lambda_1,\ldots,\lambda_k} = \sum_{j=1}^{k} (-1)^j \sigma_{\lambda_1,\ldots,\lambda_{j-1},\lambda_{j+1}-1,\ldots,\lambda_k-1} \cdot \sigma_{\lambda_j+k-j}$$

Using this relation and induction obtain the following formula for the class of any Schubert cycle in terms of special Schubert cycles.

Theorem 2.12 (Giambelli's formula)**.** *Any Schubert cycle may be expressed as a linear combination of products of special Schubert cycles as follows*

$$\sigma_{\lambda_1,\ldots,\lambda_k} = \begin{vmatrix} \sigma_{\lambda_1} & \sigma_{\lambda_1+1} & \sigma_{\lambda_1+2} & \cdots & \sigma_{\lambda_1+k-1} \\ \sigma_{\lambda_2-1} & \sigma_{\lambda_2} & \sigma_{\lambda_2+1} & \cdots & \sigma_{\lambda_2+k-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \sigma_{\lambda_k-k+1} & \sigma_{\lambda_k-k+2} & \sigma_{\lambda_k-k+3} & \cdots & \sigma_{\lambda_k} \end{vmatrix}$$

Exercise 2.13. Use Giambelli's formula to express $\sigma_{3,2,1}$ in $G(4,8)$ in terms of special Schubert cycles. Using Pieri's rule find the class of its square.

Pieri's formula and Giambelli's formula together give an algorithm for computing the cup product of any two Schubert cycles. Unfortunately, in practice this algorithm is hard to implement. We will rectify this problem shortly.

So far we have treated the Grassmannian simply as a complex manifold. For the sake of completeness, we recall how to endow it with the structure of a smooth, projective variety. Using the Plücker coordinates we can embed $G(k,V)$ into $\mathbb{P}(\bigwedge^k V)$. Given a k-plane Ω we can choose a basis for it $v_1, \ldots, v_k$. Then we can define the map $Pl : G(k,n) \to \mathbb{P}(\bigwedge^k V)$ by sending the k-plane Ω to $v_1 \wedge \cdots \wedge v_k$. A change of basis changes the image by the determinant of the matrix giving the change of basis. Hence the map is well-defined as a point of $\mathbb{P}(\bigwedge^k V)$.

The map is injective since we can recover Ω from its image $p = [v_1 \wedge \cdots \wedge v_k] \in \mathbb{P}(\bigwedge^k V)$ as the set of all vectors $v \in V$ such that $v \wedge v_1 \wedge \cdots \wedge v_k = 0$. A point of $\mathbb{P}(\bigwedge^k V)$ is in the image of this map if and only if the representative $\sum p_{i_1,\ldots,i_k} e_1 \wedge \cdots \wedge e_{i_k}$ is completely decomposable. It is not hard to characterize the subvariety of $\mathbb{P}(\bigwedge^k V)$ corresponding to completely decomposable elements. An element $x \in \bigwedge^k V$ is completely decomposable if and only if $< u, x > \wedge x = 0$ for every $u \in \bigwedge^{k-1} V^*$. Writing this in coordinates we obtain the Plücker relations

$$\sum_{s=1}^{k+1} (-1)^s p_{i_1,\ldots,i_{r-1},j_t} p_{j_1,\ldots,\hat{j_t},\ldots j_{r+1}} = 0.$$

These Plücker relations generate the ideal of the Grassmannian.

Everyone's favorite example is $G(2,4)$. In that case there is a unique Plücker relation

$$p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} = 0.$$

Hence the Plücker map embeds $G(2,4)$ in $\mathbb{P}^5$ as a smooth quadric hypersurface.

Exercise 2.14. Show that the locus where a Plücker coordinate vanishes corresponds to a Schubert variety Σ_1. Observe that the class of Σ_1 generates the second homology of the Grassmannian. In particular, the Picard group is isomorphic to $\mathbb{Z}$. Conclude that $\mathcal{O}_{G(k,n)}(\Sigma_1)$ is the very ample generator of the Picard group and it gives rise to the Plücker embedding.

We can compute the degree of the Grassmannian $G(k,n)$ under the Plücker embedding. The answer is provided by $\sigma_1^{k(n-k)}$. When $k = 2$, this computation is relatively easy to carry out. By Pieri's formula σ_1 times any cycle in $G(2,n)$ either increases the first index of the cycle or it increases the second index provided that it is less than the first index. This means that the degree of the Grassmannian $G(2,n)$ is the number of ways of walking from one corner of an $(n-2) \times (n-2)$ to the opposite corner without crossing the diagonal. This is well-known to be the Catalan number

$$\frac{(2(n-2))!}{(n-2)!(n-1)!}.$$

The general formula is more involved. The degree of $G(k,n)$ is given by

$$(k(n-k))! \prod_{i=1}^{k} \frac{(i-1)!}{(n-k+i-1)!}.$$

The local structure of the Grassmannian. The tangent bundle of the Grassmannian has a simple intrinsic description in terms of the tautological bundle T and the quotient bundle Q. There is a natural identification of the tangent bundle of the Grassmannian with homomorphisms from T to Q, in other words

$$TG(k,n) = \mathrm{Hom}(T,Q).$$

In particular, the tangent space to the Grassmannian at a point $[\Omega]$ is given by $\mathrm{Hom}(\Omega, V/\Omega)$. One way to realize this identification is to note that the Grassmannian is a homogeneous space for $GL(n)$. The tangent space at a point may be naturally identified with quotient of the Lie algebra of $GL(n)$ by the Lie algebra of the stabilizer. The Lie algebra of $GL(n)$ is the endomorphisms of V. Those that stabilize Ω are those homomorphisms $\phi : V \to V$ such that $\phi(\Omega) \subset \Omega$. These homomorphisms are precisely homomorphisms $\mathrm{Hom}(\Omega, V/\Omega)$.

Exercise 2.15. Use the above description to obtain a description of the tangent space to the Schubert variety $\Sigma_{\lambda_1,\ldots,\lambda_k}$ at a smooth point $[\Omega]$ of the variety.

We can use the description of the tangent space to check that the intersection of Schubert cycles in previous calculations were indeed transverse. For example, suppose we take the intersection of two Schubert varieties Σ_1 in $\mathbb{G}(1,3)$ defined with respect to two skew-lines. Then the intersection is a smooth variety. In vector space notation, we can assume that the conditions are imposed by two non-intersecting two-dimensional vector spaces V_1 and V_2. Suppose a 2-dimensional vector space Ω meets each in dimension 1. The tangent space to Ω at the intersection is given by

$$\phi \in \mathrm{Hom}(\Omega, V/\Omega) \text{ such that } \phi(\Omega \cap V_i) \subset [V_i] \in V/\Omega.$$

As long as V_1 and V_2 do not intersect, Ω has exactly a one-dimensional intersection with each of V_i and these span Ω. On the other hand, the quotient of V_i in V/Ω is one-dimensional. We conclude that the dimension of such homomorphisms is 2. Since this is equal to the dimension of the variety, we deduce that the variety is smooth.

Exercise 2.16. Carry out a similar analysis for the other examples we did above.

Definition 2.17. Let S be a scheme, E a vector bundle on S and k a natural number less than or equal to the rank of E. The functor

$$Gr(k,E) : \{\text{schemes over} S\} \to \{\text{sets}\}$$

associates to every S scheme X the set of rank k subvector bundles of $E \times_S X$.

Theorem 2.18. *The functor $Gr(k,E)$ is represented by a scheme $G_S(k,E)$ and a subvector bundle $U \subset E \times_S G_S(k,E)$ of rank k.*

3. A Littlewood-Richardson rule

Positive combinatorial rules for determining Littlewood - Richardson coefficients are known as Littlewood - Richardson rules. As an introduction to the degeneration techniques that we will employ through out this course, we give a Littlewood - Richardson rule for the Grassmannian.

There are many Littlewood - Richardson rules for the Grassmannian. You can find other Littlewood - Richardson rules in [Ful1], [V1], [KT]. The rule we will develop here is a geometric Littlewood - Richardson rule. These rules have many applications in geometry. For some examples of applications to positive characteristic, Schubert calculus over $\mathbb{R}$ and monodromy groups see [V2].

The fundamental example. Consider calculating σ_1^2 in $G(2,4)$. Geometrically we would like to calculate the class of two dimensional linear spaces that meet two general two dimensional linear spaces in a four dimensional vector space. Projectivizing this question is equivalent to asking for the class of lines in $\mathbb{P}^3$ intersecting two general lines.

The idea underlying the approach to answering this question is classical. While it is hard to see the Schubert cycles that constitute this intersection when the two lines that define the two Schubert cycles are general, the result becomes easier if the lines are in special position.

To put the lines l_1 and l_2 in a special position fix a plane containing l_1 and rotate it about a point on it, so that it intersects l_2. As long as l_1 and l_2 do not intersect, they are in general position since the automorphism group of $\mathbb{P}^3$ acts transitively on pairs of skew lines. However, when l_1 and l_2 intersect, then they are no longer in general position.

We can ask the following fundamental question: What is the limiting position of the lines that intersect both l_1 and l_2? Since intersecting the lines are closed conditions, any limit line has to continue to intersect l_1 and l_2. There are two ways that a line can intersect two intersecting lines in $\mathbb{P}^3$. Either the line passes through their intersection point, or if it does not pass through its intersection point then it must lie in the plane spanned by the two lines. Note that these are both Schubert cycles. Since their dimensions are equal to the dimension of the original variety, the class of the original variety has to be the sum of multiples of these two Schubert cycles.

We can determine that the multiplicities are one as follows. The tangent space to the Grassmannian $G(k, n)$ at a point Λ is given by It suffices then to check that the two cycles intersect transversely at a general point of each of the Schubert cycles.

A *Mondrian tableau* associated to a Schubert class $\sigma_{\lambda_1,\cdots,\lambda_k}$ in $G(k, n)$ is a collection of k nested squares labeled by integers $1, \dots, k$ where the j-th box has size $n-k+j-\lambda_j$ and a box of smaller index is contained in every box of larger index. Figure 1 depicts a Mondrian tableau for $\sigma_{2,1}$ in $G(3, 6)$.

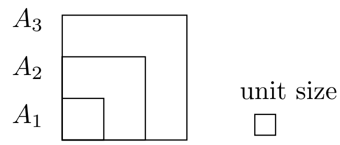

FIGURE 1. The Mondrian tableau associated to $\sigma_{2,1}$ in $G(3, 6)$.

In Mondrian tableaux a box of side length s denotes a vector space of dimension s. If a box S_1 is contained in another box S_2, then the linear space represented by S_1 is a subspace of the linear space represented by S_2. The reader should think of unit squares along the anti-diagonal as giving a basis of the underlying vector space. The vector space represented by a box is the span of the basis elements it contains. In a Mondrian tableau associated to σ_λ the k-plane is required to meet the vector space represented by a box in dimension equal to the number of boxes contained in that box (including itself). We will denote boxes in a Mondrian tableau by capital letters in the math font (e.g. A_i) and the vector spaces they represent by the corresponding letter in Roman font (e.g. A_i).

We stress that any nested sequence of a boxes that have their centers along the anti-diagonal defines a Schubert cycle. The boxes need not be left or right aligned.

The game. To multiply two Schubert classes σ_λ and σ_μ in $G(k, n)$ we place the tableau associated to λ starting from the lower left hand corner and the tableau associated to μ starting from the upper right hand corner of an $n \times n$ square. The squares in the λ (μ) tableau are all left (respectively, right) aligned with respect to the $n \times n$ square. We will denote the boxes corresponding to λ and μ by A_i and B_j, respectively. The left panel in Figure 2 shows the initial tableau for the multiplication $\sigma_{2,1,1} \cdot \sigma_{1,1,1}$ in $G(3, 6)$.

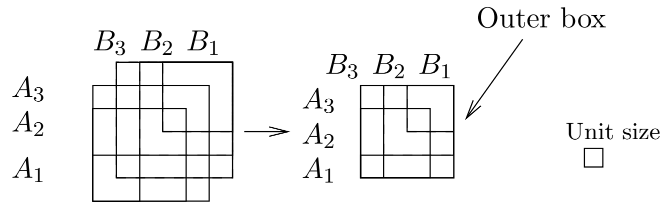

FIGURE 2. An application of the OB rule.

Initially the two Schubert cycles are defined with respect to two transverse flags. If the intersection of the two Schubert cycles is non-empty, then the Schubert cycles have to satisfy certain conditions. A preliminary rule (MM rule) guarantees that these conditions are satisfied. Then there are some simplifications that reduce the problem to a smaller problem. The OB and S rules give these simplifications.

• **The MM rule.** We check that A_i intersects B_{k-i+1} in a square of side length at least one for every i between 1 and k. If not, we stop. The Schubert cycles have empty intersection. In other words, the class of the intersection is zero.

In a k-dimensional vector space V^k every i-dimensional subspace (such as $V^k \cap \mathrm{A}_i$) **M**ust **M**eet every $k-i+1$-dimensional subspace (such as $V^k \cap \mathrm{B}_{k-i+1}$) in at least a line. The intersection of two Schubert cycles is zero if and only if the initial tableau formed by the two cycles does not satisfy the MM rule.

• **The OB rule.** We call the intersection of A_k and B_k the **O**uter **B**ox of the tableau. We replace every square with its intersection with the outer box.

Since the k-planes are contained in both A_k and B_k, they must be contained in their intersection. Figure 2 shows an example in $G(3,6)$.

• **The S rule.** We check that A_i and B_{a-i} touch or have a common square. If not, we remove the rows and columns between these squares as shown in Figure 3.

This rule corresponds to the fact that an a-dimensional vector space lies in the **S**pan of any two of its subspaces of complementary dimension whose only intersection is the origin. This rule removes any basis element of V that is not needed in expressing the a-planes parameterized by the intersection of the two Schubert varieties.

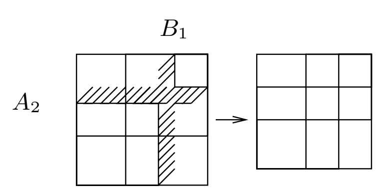

FIGURE 3. Adjusting the span of the linear constraints.

Once we have performed these preliminary steps, we will inductively build a new flag (the D flag) by degenerating the two flags (the A and B flags). At each stage of the game we will have a partially built new flag (depicted by D boxes that arise as intersections of A and B boxes) and partially remaining A and B flags (depicted by boxes $A_i, \ldots, A_a$ and B_k, $B_{k-i}, \ldots, B_1$). After nesting the D boxes, we will increase the dimension of the intersection of A_i with B_{k-i} by one in order of increasing i. We will depict this move in the Mondrian tableau by sliding A_i anti-diagonally up by one unit. Assuming that there are no boxes left justified with A_i, the corresponding degeneration can be described as follows:

Let s be the side-length of A_i and suppose that initially A_i and B_{a-i} intersect in a square of side-length r. There is a family of s-dimensional linear spaces $\mathrm{A}_i(t)$ parameterized by an open subset $0 \in U \subset \mathbb{P}^1$ such that over the points $t \in U$ with $t \neq 0$, the dimension of intersection $\mathrm{A}_i(t) \cap \mathrm{B}_{k-i}$ is equal to r and when $t = 0$, the dimension of intersection $\mathrm{A}_i(0) \cap \mathrm{B}_{a-i}$ is $r + 1$. Denoting the basis vectors represented by the unit squares along the diagonal by $e_1, \ldots, e_n$, we explicitly take the family to be

$$\mathrm{A}_i(t) = \text{ the span of } \{(te_1 + (1-t)e_{s+1}, e_2, \ldots, e_s\}.$$

When $t = 1$, we have our original vector space A_i represented by the old position of the box A_i. When $t = 0$, we have the new vector space $\mathrm{A}_i(0)$ represented by the new position of the box A_i. When $t = 0$, the intersection of Schubert varieties defined with respect to the A and B flags either remains irreducible or breaks into two irreducible components. The LR rule records these possibilities and can be informally phrased as:

If the a-planes in the limit do not intersect $\mathrm{A}_i(0) \cap \mathrm{B}_{k-i}$*, then they must be contained in their new span.*

The main work in establishing the rule rests in describing which varieties (equivalently which Mondrian tableaux) occur as a result of the degenerations (equivalently moves). Very generally we can define a Mondrian tableau in $G(k, n)$ to be a collection of k boxes contained in an $n \times n$ box satisfying the following two properties:

(1) None of the boxes are equal to the span of the boxes contained in them.
(2) Let S_1 and S_2 be any two boxes in the tableau. If the number of boxes contained in their span but not contained in S_1 is r, then the side length of S_1 is at least r less than the side length of their span.

We can associate an irreducible subvariety of the Grassmannian $G(k, n)$ to such a tableau. We first define an open subset of the variety by requiring the k-planes to meet the vector spaces represented by each box in dimension equal to the number of boxes contained in that box (including itself). We further require the vector subspaces of the k-planes contained in the vector spaces represented by any two boxes to only meet along the subspaces contained in subspaces contained in boxes common to both of the boxes and otherwise to be independent. The variety associated to the generalized Mondrian tableau is the closure in $G(k, n)$ of the quasi-projective variety parameterizing such k-planes.

The intersection of two Schubert varieties can be turned to such a tableau by replacing the boxes A_i and B_j by the boxes consisting of the intersections $A_i \cap B_{k-i+1}$. Here we will not discuss the rule that expresses the classes of the varieties defined by these very general tableaux as a sum of Schubert varieties. When we resolve the intersection of Schubert varieties into a union of Schubert varieties, only very few of these varieties occur. During the game the Mondrian tableaux that occur have more structure. The admissible tableaux characterize the ones that occur.

A Mondrian tableau is *admissible* for $G(k,n)$ if the squares that constitute the tableau (except for the outer box) are uniquely labeled as an indexed A, B or D box such that

(1) The boxes $A_k = B_k$ form the outer box. They have side length $m \leq n$ and contain the entire tableau.
(2) The A boxes are all nested, distinct, left aligned and strictly contain all the D boxes. If the number of D boxes is $i-1<k$, then the A boxes are $A_i, A_{i+1}, \ldots, A_k$ with the smaller index corresponding to the smaller box. (In particular, the number of A boxes is $k-i+1$, hence the number of A and D boxes add up to k.)
(3) The B boxes are all nested, distinct and right aligned. They are labeled $B_k, B_{k-i}, B_{k-i-1}, \ldots, B_1$, where a smaller box has the smaller index. (In particular, the number of B boxes equals the number of A boxes.) The A and B boxes satisfy the MM and S rules. The D boxes may intersect B_{k-i}, but none are contained in B_{k-i}. The side length of B_{k-i} is at least i units smaller than the side length of the outer box and at least h units smaller than the side length of the box spanned by the boxes D_s for $1 \leq s \leq h$ and B_{k-i} for every $1 \leq h \leq i-1$.
(4) The D boxes are labeled $D_1, \ldots, D_{i-1}$. They do not need to be nested; however, there can be at most one unnested D box. An unnested D box is defined to be a D box that does not contain every D box of smaller index. More precisely, if D_j does not contain all the D boxes of smaller index, then it does not contain any of the D boxes of smaller index; it is contained in every D box of larger index; and $D_h \subset D_k$ for every $h<k$ as long as h and k are different from j. All the D boxes of index lower than j are to the lower left of D_j. D_{j-1} and D_j share a common square or corner. If there is an unnested D box D_j, the D or A box with one larger index is at least one larger than the span of the D boxes contained in it. The side length of D_j is at least i smaller than the side length of the square spanned by D_i and D_j for every $i<j$.

Given an admissible Mondrian tableau there is a corresponding subvariety of $G(k,n)$. The corresponding subvariety is defined as the closure of the locus of k-planes that satisfy certain numerical conditions with respect to the vector spaces represented by the A, B and D boxes. Precisely, the variety associated to an admissible Mondrian tableau is the closure of the locus of k-planes that intersect the vector spaces represented by the boxes D_s, $s=1,\ldots,i-1$, A_t, $t=i,\ldots,k$, and B_u, $u=k-i,\ldots,1$, in dimension equal to the number of boxes contained in them. This defines an irreducible variety. The strategy is to specialize the flags in order to break such a variety into a union of two varieties that have the same form. The moves on the Mondrian tableaux achieve this purpose.

Let M be an admissible Mondrian tableau with an outer box of side length m. If all the D boxes are nested, we slide the smallest A box A_i anti-diagonally up by one unit. Any of the D boxes that touch the lower left hand corner of A_i move one unit up with A_i. The remaining D boxes do not move. If the side length of A_{i+1} is not one larger than the side length of A_i or the side length of B_{k-i} is not

$m - i$ (informally, if A_i or B_{k-i} are not as large as possible given A_{i+1} and B_k), we replace M with the two tableaux described in Possibilities 1 and 2. If the side length of A_{i+1} is one larger than the side length of A_i or the side length of B_{k-i} is $m - i$, we replace M only with the tableau in Possibility 1.

- **Possibility 1.** We delete A_i and B_{k-i} and replace them with D_i which is the new intersection of A_i and B_{k-i}. If D_i does not intersect or touch B_{k-i-1}, we slide all the D boxes anti-diagonally up until D_i touches B_{k-i-1}. All the remaining boxes stay as in M.

- **Possibility 2.** We shrink the outer box by one so that it passes along the new boundary of A_i and B_{k-i} and we delete the column and row that lies outside this box. The rest of the boxes stay as in M.

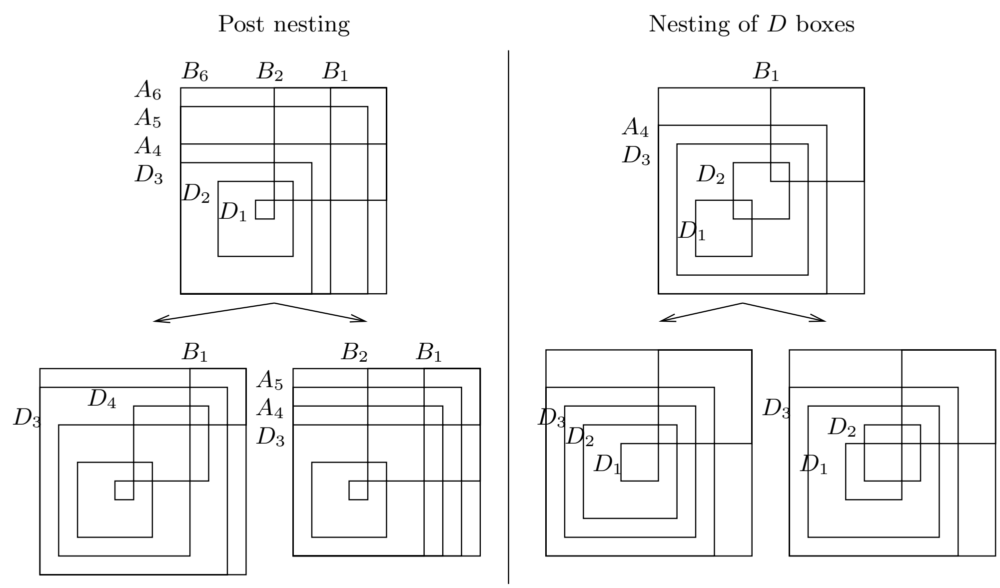

FIGURE 4. Admissible Mondrian tableaux and the moves.

The two tableaux obtained from M are depicted in Figure 4. Geometrically in the first possibility the k-plane intersects the new intersection $\mathrm{A}_i \cap \mathrm{B}_{k-i}$. In the second possibility, the k-plane lies in the new span of A_i and B_{k-i}.

Nesting the D boxes. Now suppose that there is an unnested D box D_j. Assume that the smallest square containing D_{j-1} and D_j has side length d_j. In this case we move D_{j-1} anti-diagonally up by one unit. Any D boxes contained in D_{j-1} and left justified with it move one unit up with D_{j-1}. The remaining boxes stay fixed. If the side length of D_j is less than $d_j - j + 1$ or after the move D_{j-1} does not contain D_j, we replace the tableau M with the following two tableaux. If the side length of D_j is $d_j - j + 1$ or after the move D_{j-1} contains D_j, we replace M only with the tableau in Possibility 1.

- **Possibility 1.** We delete D_j and D_{j-1}. We draw the old span and label it D_j. We also draw the new intersection and label it D_{j-1}. If D_{j-1} does not meet or touch B_{a-i}, we slide all the D boxes of index at most $j-1$ anti-diagonally up until it does. We keep the remaining boxes as in M.

• **Possibility 2.** We place D_{j-1} in its new position and keep all the remaining boxes as in M.

It is not hard to check that the results of the moves transform an admissible Mondrian tableau to one or two new admissible Mondrian tableaux. Therefore, we can continue applying the moves to each of the resulting tableaux. After a cycle of moves the number of A and B boxes decrease and the number of nested D boxes increases. Eventually all the boxes will be nested again. The corresponding variety is a Schubert variety. If we apply the moves to each of the Mondrian tableaux that occur until all the boxes are nested, we end up with a collection of tableaux corresponding to Schubert varieties.

A dimension calculation shows that applying the degeneration described above to the variety represented by an admissible Mondrian tableau results in the varieties represented by the Mondrian tableaux described in the possibilities. A multiplicity calculation shows that each of the varieties occur with multiplicity one. The following theorem is a consequence of these calculations.

Theorem 3.1. *The LR coefficient $c^{\nu}_{\lambda,\mu}$ of $G(k,n)$ equals the number of times σ_ν results in a game of Mondrian tableaux starting with σ_λ and σ_μ in an $n \times n$ box.*

We conclude the discussion of the geometric Littlewood - Richardson rules for the ordinary Grassmannian with an example. We compute $\sigma_{2,1}^2$ in $G(3,6)$ (see Figure 5). We start by moving the smallest A box. There are two possibilities. We replace the tableau by the two tableaux where we take the intersection of A_1 and B_2 (and slide it up) and keep everything else the same and where we restrict the tableau to the new span of A_1 and B_2. We continue resolving the first tableau by moving A_2. Again there are two possibilities. In the second tableau B_2 is as large as possible given the outer box, so when we move A_1, there is only one possibility. We then move A_2 and now there are two possibilities. We replace the tableau with the tableau where we take the intersection of A_2 and B_1 and with the tableau where we restrict the tableau to the new span of A_2 and B_1. Continuing we conclude that

$$\sigma_{2,1}^2 = \sigma_{3,3} + 2\sigma_{3,2,1} + \sigma_{2,2,2}.$$

Exercise 3.2. Show that when one takes one of the Schubert cycles to be a special Schubert cycle, one recovers Pieri's rule. Our proof of the Littlewood - Richardson rule used Pieri's rule. Carry out the multiplicity calculations explicitly for that case to reprove Pieri's rule.

Exercise 3.3. Formulate and prove a Littlewood - Richardson rule that decomposes the class of any variety described by a generalized Mondrian tableau into a sum of Schubert classes.

Exercise 3.4. Using the rule compute the Littlewood - Richardson coefficients of small Grassmannians.

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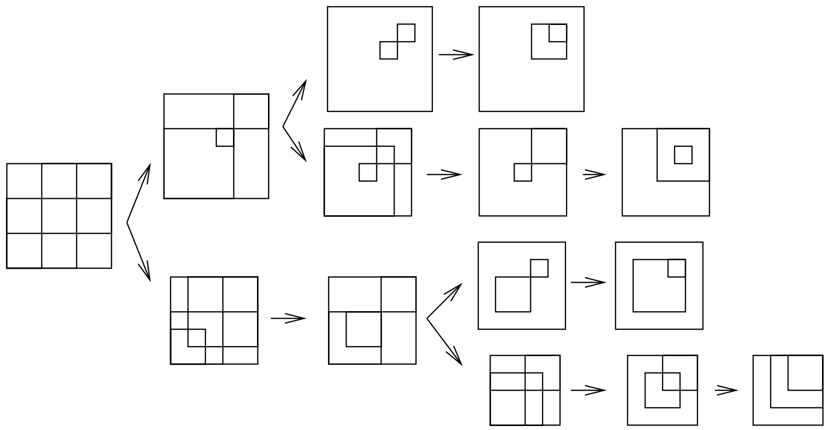

FIGURE 5. The product $\sigma_{2,1}^2 = \sigma_{3,3} + 2\sigma_{3,2,1} + \sigma_{2,2,2}$ in $G(3,6)$.

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