DIVISOR CLASSES ON THE MODULI SPACE OF CURVES

1. The cohomology of the moduli space of pointed genus zero curves

In this section we discuss the Chow rings of the moduli spaces of n-pointed genus zero curve $\overline{\mathrm{M}}_{0,n}$. Recall that we are working over the complex numbers $\mathbb{C}$. The cohomology and Chow groups of $\overline{\mathrm{M}}_{0,n}$ turn out to be isomorphic. The main statement is that the Chow/cohomology ring of $\overline{\mathrm{M}}_{0,n}$ is generated by the classes of boundary divisors. The main reference for this section is [Kee].

The basic strategy for determining the Chow/cohomology ring of $\overline{\mathrm{M}}_{0,n}$ is to exhibit $\overline{\mathrm{M}}_{0,n}$ as a sequence of blow-ups of the product $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ of $n-3$ copies of $\mathbb{P}^1$ along smooth centers. One then inductively calculates the Chow ring at each stage of the blow-up process using the following basic theorem.

Theorem 1.1 (The Chow ring of blow-ups)**.** *Let X be a codimension d smooth subvariety of a smooth variety Y with normal bundle $N_{X/Y}$. Let $i : X \to Y$ denote the inclusion of X in Y. Let $\tilde{Y}$ be the blow-up of Y along X. Assume that*

$$i^* : A(Y) \to A(X)$$

is surjective. Then

$$A^*(\tilde{Y}) \cong \frac{A^*(Y)[\zeta]}{< \zeta \ker(i^*), \zeta^d + \zeta^{d-1}c_1(N_{X/Y}) + \cdots + \zeta c_{d-1}(N_{X/Y}) + c_d(N_{X/Y}) >}$$

where $-\zeta$ is the class of the exceptional divisor.

Now we introduce the generators of the Chow ring. Let S be a subset of $\{1, \dots, n\}$ with the property that both S and its complement have at least two elements. We will denote the number of elements of S by $\#S$. Given such a set we can define the class δ_S on $\overline{\mathrm{M}}_{0,n}$ as the class of the divisor Δ_S of stable curves C that have a separating node that divides C into $C_1 \cup C_2$ where the labelings of the points on C_1 are precisely the elements of S and the labelings of the points on C_2 are precisely the elements of S^c. The divisor Δ_S is a normal crossings divisor isomorphic to

$$\Delta_S \cong \overline{M}_{0,S\cup\{r\}} \times \overline{M}_{0,S^c\cup\{s\}}$$

obtained by the map that glues the marked points r and s.

The main theorem about the Chow ring of $\overline{\mathrm{M}}_{0,n}$ is the following:

Theorem 1.2 (Keel)**.** *The Chow/cohomology ring of $\overline{\mathrm{M}}_{0,n}$ is generated by the classes δ_S where $\#S \geq 2$ and $\#S^c \geq 2$ subject to the following relations:*

(1) $\delta_S = \delta_{S^c}$.

(2) *For any four distinct elements $i, j, k, l \in \{1, \dots, n\}$*

$$\sum_{i,j\in S,k,l\notin S} \Delta_S = \sum_{i,k\in S,j,l\notin S} \delta_S = \sum_{i,l\in S,j,k\notin S} \delta_S.$$

(3) *For two subsets S and T*

$$\delta_S \delta_T = 0$$

unless $S \subset T, T \subset S, S \subset T^c$ or $T^c \subset S$.

Example 1.3. Since $\overline{\mathrm{M}}_{0,4} \cong \mathbb{P}^1$, the classes of the three boundary divisors $\Delta_{\{1,2\}}$, $\Delta_{\{1,3\}}$ and $\Delta_{\{1,4\}}$ are linearly equivalent. If we specialize the statement of the theorem to $n = 4$ we recover the cohomology of $\mathbb{P}^1$.

Remark 1.4. It is easy to see that the claimed relations are satisfied. The divisor classes δ_S and δ_{S^c} are equal since the divisors they represent are equal.

To prove the relation

$$\sum_{i,j \in S, k,l \notin S} \delta_S = \sum_{i,k \in S, j,l \notin S} \delta_S$$

consider the map

$$\pi_{i,j,k,l} : \overline{\mathrm{M}}_{0,n} \to \overline{\mathrm{M}}_{0,4}$$

given by forgetting all the points, but the points labeled by i, j, k, l and stabilizing the resulting curve. The pull-back of the divisor class $\delta_{\{i,j\}}$ on $\overline{\mathrm{M}}_{0,4}$ is given by

$$\sum_{i,j \in S, k,l \notin S} \delta_S.$$

The pull-back of the divisor class $\delta_{\{i,k\}}$ on $\overline{\mathrm{M}}_{0,4}$ is given by

$$\sum_{i,k \in S, j,l \notin S} \delta_S.$$

Since these divisors have to be linearly equivalent, the relation follows.

Finally to see that $\delta_S \delta_T = 0$ unless $S \subset T, T \subset S, S \subset T^c$ or $T^c \subset S$ note that two divisors Δ_S and Δ_T contain the point represeting a curve C in their intersection if and only if there are two nodes on C that divide C into C_1, C_2 and C_1', C_2' where the labeling on C_1 is S and the labeling on C_1' is T. Observe that unless the conditions $S \subset T, T \subset S, S \subset T^c$ or $T^c \subset S$ are satisfied Δ_S and Δ_T are disjoint, hence the product of their classes is zero in the Chow/cohomology ring.

Example 1.5. We can view $\overline{\mathrm{M}}_{0,5}$ as the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at the three points $(0,0), (1,1)$ and (∞, ∞). Hence $\overline{\mathrm{M}}_{0,5}$ is isomorphic to the Del Pezzo surface D_5. The 10 boundary divisors on $\overline{\mathrm{M}}_{0,5}$ correspond to the 10 exceptional curves on D_5. We can recover the cohomology ring of D_5 from Keel's relations. Note that Keel's second set of relations in this case give us that for any distinct 4-tuple i, j, k, l:

$$\delta_{i,j} + \delta_{k,l} = \delta_{i,k} + \delta_{j,l} = \delta_{i,l} + \delta_{j,k}.$$

Multiplying these relations by $\delta_{i,j}$ and using the third set of relations easily gives that $\delta_{i,j}^2 = \delta_{k,l}^2 = -\delta_{r,s}\delta_{t,u}$ for any i, j, k, l and distinct r, s, t, u. Finally all triple products vanish. Note that one can give a very simple presentation of the cohomology ring of D_5 realizing it as the blow-up of $\mathbb{P}^2$ in four points. Sending the divisors $\delta_{i,5}$ to the classes of the four exceptional divisors $E_1, \dots, E_4$ and $\delta_{i,j}$ to $H - E_k - E_l$ (where $\{k, l\}$ is disjoint from $\{i, j, 5\}$) for the remaining i, j gives a ring isomorphism. Here H denotes the hyperplane class on $\mathbb{P}^2$. Hence, Keel's presentation is not necessarily the simplest presentation.

Exercise 1.6. Verify the claims made in the discussion of the previous example.

Exercise 1.7. Using the description of the cohomology ring of $\overline{\mathrm{M}}_{0,n}$ determine its Betti numbers. Find the Euler characteristic of $\overline{\mathrm{M}}_{0,n}$.

Now we can describe the main technical tool that allows one to compute the Chow ring of $\overline{\mathrm{M}}_{0,n}$. Consider the map

$$\pi_{n+1} : \overline{\mathrm{M}}_{0,n+1} \to \overline{\mathrm{M}}_{0,n}$$

given by forgetting the last marked point. This morphism factors through

$$\begin{tikzcd} \overline{\mathrm{M}}_{0,n+1} \arrow[r, "\phi"'] \arrow[d, "\pi_{n+1}"'] & \overline{\mathrm{M}}_{0,n} \times \overline{\mathrm{M}}_{0,4} \arrow[d, "pr_1"'] \\ \overline{\mathrm{M}}_{0,n} \arrow[r, "id"'] & \overline{\mathrm{M}}_{0,n} \end{tikzcd}$$

where pr_2 is the projection onto the second factor and ϕ is induced by $(\pi_{n+1}, \pi_{4,\dots,n})$ where $\pi_{4,\dots,n}$ is the morphism that forgets all but the points marked $1, 2, 3, n+1$. The calculation is based on the obervation that the morphism ϕ is in fact a sequence of $n-3$ blow-ups along explicit smooth centers.

Set $X_1 = \overline{\mathrm{M}}_{0,n} \times \overline{\mathrm{M}}_{0,4}$. If S is a subset of $\{1, \dots, n\}$, we can embed the divisors Δ_S into X_1 by first mapping Δ_S by the universal section corresponding to the i-th point to $\overline{\mathrm{M}}_{0,n+1}$, then following it with the map to X_1. Let X_2 be the blow-up of X_1 along Δ_S where $\#S^c = 2$ and S contains at most one of $1, 2, 3$. Note that these are disjoint in X_1. Let X_3 be the blow-up of X_2 along the proper transform of the Δ_S with $\#S^c = 3$ and S contains at most one of $1, 2, 3$. Continue in this way where X_k is the blow-up of X_{k-1} along the proper transform of Δ_S with $\#S^c = k$ such that S contains at most one of $1, 2, 3$. Then $\overline{\mathrm{M}}_{0,n+1} \cong X_{n-2}$ and the map

$$\phi : \overline{\mathrm{M}}_{0,n+1} \to \overline{\mathrm{M}}_{0,n} \times \overline{\mathrm{M}}_{0,4}$$

is the blowing-up just described.

To finish the proof of Theorem 1.2 we simply have to inductively apply the theorem describing the Chow ring of the blow-up repeatedly. This is messy but straightforward (see [Kee]).

Remark 1.8. Note that the construction of $\overline{\mathrm{M}}_{0,n}$ as a blow-up of $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ implies that the Chow ring and the cohomology ring are isomorphic. In particular, $\overline{\mathrm{M}}_{0,n}$ does not have any odd cohomology.

Remark 1.9. Observe that $\mathrm{M}_{0,n}$ is an affine variety. Fixing three of the points at $0, 1$ and ∞ we can view this space as the complement of hyperplanes in $\mathbb{C}^{n-3}$. Hence, $\mathrm{M}_{0,n}$ is affine of dimension $n-3$. Recall that the homology of an affine manifold vanishes above half its real dimension.

Theorem 1.10. *Let X be a smooth, complex affine variety of complex dimension n, then $H_k(X, \mathbb{Q}) = 0$ for $k > n$.*

Milnor's proof of this theorem using Morse theory is one of the most beautiful proofs in mathematics (see [Mi]). We conclude that the cohomology $H_k(\mathrm{M}_{0,n}, \mathbb{Q}$ vanishes for $k > n-3$.

Note that Theorem 1.2 in particular determines second homology/the Picard group of $\overline{\mathrm{M}}_{0,n}$.

Corollary 1.11. *The Picard group of $\overline{\mathrm{M}}_{0,n}$ is generated by the classes of boundary divisors Δ_S subject to the relations $\delta_S = \delta_{S^c}$ and for any four distinct elements $i, j, k, l \in \{1, \ldots, n\}$*

$$\sum_{i,j \in S, k,l \notin S} \delta_S = \sum_{i,k \in S, j,l \notin S} \delta_S = \sum_{i,l \in S, j,k \notin S} \delta_S.$$

Exercise 1.12. Determine the class of the canonical divisor of $\overline{\mathrm{M}}_{0,n}$.

In fact, we can do better than the previous corollary.

Proposition 1.13. *Let $n \geq 4$. Fix three distinct indeces i, j, k. The second cohomology group of $\overline{\mathrm{M}}_{0,n}$ has basis $\delta_{\{j,k\}}$, δ_S where $i \in S$ and $\#S \leq n-3$.*

Proof. We can give an elementary proof of this result that does not depend on the complicated combinatorics of Keel's theorem. We already saw that the boundary divisors generate the second cohomology (e.g. the complement of the boundary is $\mathbb{C}^{n-3}$ with some hyperplanes removed) and that the relations in Keel's theorem are satisfied. We need to show that we can express all boundary divisors in terms of these and that these are independent. The only boundary divisors not on the list are those of the form $\delta_{u,v}$ where neither of u, v is i and the pair is not j, k. Writing the boundary relation for i, w, u, v we see that

$$\delta_{\{i,w\}} + \sum_{i,w \in S, u,v \notin S, 3 \leq \#S \leq n-3} \delta_S + \delta_{\{u,v\}} = \delta_{\{i,v\}} + \sum_{i,v \in S, u,w \notin S, 3 \leq \#S \leq n-3} \delta_S + \delta_{\{u,w\}}.$$

Hence $\delta_{\{u,v\}} = \delta_{\{u,w\}}$. Taking $v = j$ and then applying the relation again to replace u by k, we see that the given boundary divisors generate.

We prove the fact that they are independent by induction. Suppose there was a relation among them. Look at the morphism forgetting a point other than i, j, k. It immediately follows that all the coefficients of the relation have to be zero. ☐

Remark 1.14. Note that the following proposition implies that the rank of the second cohomology group is

$$2^{n-1} - \frac{n^2 - n + 2}{2}.$$

2. The second homology group of the moduli space of curves

Originally Harer determined the second homology group of the moduli space of curves by computing the second homology group of the mapping class group. Some good references for Harer's work on this computation is Harer's original paper [Har1] and Harer's C.I.M.E. notes [Har2]. Here we will outline Arbarello and Cornalba's algebraic approach to the computation of the second homology group [AC2].

We begin by introducing some divisor classes on $\overline{\mathrm{M}}_{g,n}$. Let

$$\pi_{n+1} : \overline{\mathrm{M}}_{g,n+1} \to \overline{\mathrm{M}}_{g,n}$$

denote the morphism that forgets the last marked point. Let $\omega_{\pi_{n+1}}$ be the relattive dualizing sheaf. π_{n+1} has n sections given by the marked points $p_1, \dots, p_n$. Denote the images of these sections σ_i by Σ_i. The class κ in this notation is defined by

$$\kappa = \pi_{n+1\,*}(c_1(\omega_{\pi_{n+1}}(\sum_{i=1}^{n} \Sigma_i))^2).$$

The classes of the n cotangent lines ψ_i for $1 \leq i \leq n$ are defined by

$$\psi_i = \sigma_i^*(\omega_{\pi_{n+1}}).$$

The sum $\sum_{i=1}^{n} \psi_i$ is often denoted by ψ.

Finally there are the classes of the boundary divisors. Let δ_{irr} be the class of the divisor of curves Δ_{irr} that contain a non-separating node. Let $0 \leq h \leq g$ be an integer and let S be a subset of $\{1, \dots, n\}$. Let $\delta_{h,S}$ be the class of the divisor $\Delta_{h,S}$ of curves that contain a node which separates the curve into two components of genus h with marked points p_i for $i \in S$ and genus $g-h$ and marked points p_i for $i \in S^c$. If h (respectively, $g-h$) is zero, we require that $\#S \geq 2$ ($\#S^c \geq 2$). There is one exception to this definition. When we define the class $\delta_{1,\emptyset} = \delta_{g-1,n}$, we need to be careful because a general member of this divisor has an automorphism of order 2. When we define this class, we take it to be half the class of the locus of the class of the boundary divisor. In terms of this notation the main theorem of this section is the following:

Theorem 2.1. *Let g and n be non-negative integers such that $2g-2+n > 0$ and $g > 0$. The second cohomology group $H^2(\overline{M}_{g,n}, \mathbb{Q})$ is generated by the classes κ, ψ_i for $1 \leq i \leq n$ and the classes δ_{irr} and $\delta_{h,S}$ such that $0 \leq h \leq g$ and $2h-2+\#S > 0$ and $2(g-h)-2+\#S^c > 0$.*

(1) *If $g > 2$, the relations among these classes are generated by*

$$\delta_{h,S} = \delta_{g-h,S^c}.$$

(2) *If $g = 2$, there is the additional relation*

$$5\kappa = 5\psi + \delta_{irr} - 5\delta_0 + 7\delta_1.$$

(3) *If $g = 1$, there are the following two additional relations*

$$\kappa = \psi - \delta_0, \quad 12\psi_p = \delta_{irr} + 12 \sum_{p \in S, \#S \geq 2} \delta_{0,S}.$$

Since Theorem 1.2 already determines the genus zero case we will omit it from our discussion.

The strategy of the proof is to do induction on the genus and the number of marked points. We now explain the mechanism that allows us to do this induction. Recall that since the coarse moduli scheme $\overline{\mathrm{M}}_{g,n}$ is an orbifold, Poincaré duality holds for it provided that we work with rational coefficients.

We need to know the vanishing of the k-th homology groups of $\mathrm{M}_{g,n}$ for large k. Recall Harer's theorem which states that the moduli space $M_{g,n}$ has the homotopy type of a finite cell complex of dimension $4g-4+n$ for $n > 0$. Since the homology groups of a finite cell complex vanish in dimension bigger than the dimension of the cell complex, we can deduce that

$$H_k(M_{g,n}) = 0, \quad k > 4g-4+n,\ n>0$$

Furthermore, a spectral sequence argument implies that

$$H_k(M_{g,0},\mathbb{Q}) = 0,\ k > 4g-5.$$

Combining this vanishing with Poincaré duality and the long exact sequence of cohomology

$$H^k_c(\mathrm{M}_{g,n},\mathbb{Q}) \to H^k(\overline{\mathrm{M}}_{g,n},\mathbb{Q}) \to H^k(\delta\mathrm{M}_{g,n},\mathbb{Q}) \to H^{k+1}_c(\mathrm{M}_{g,n},\mathbb{Q})$$

we conclude the following proposition.

Proposition 2.2. *The map $H^k(\overline{M}_{g,n},\mathbb{Q}) \to H^k(\delta M_{g,n},\mathbb{Q})$ is an isomorphism when $k < d(g,n)$ and injective when $k = d(g,n)$, where $d(g,n)$ is defined by*

$$d(g,n) = \begin{cases} n-4 & \text{if } g=0 \\ 2g-2 & \text{if } n=0 \\ 2g-3+n & \text{if } g,n>0. \end{cases}$$

This proposition gives us hope to do induction on the genus and the number of marked points. Recall that

$$\Delta_{irr} \cong \overline{\mathrm{M}}_{g-1,P\cup\{r,s\}}$$

where the isomorphism is obtained by attaching the marked points r and s to obtain a curve of arithmetic genus g. Similarly

$$\Delta_{h,S} \cong \overline{\mathrm{M}}_{h,S\cup\{r\}} \times \overline{\mathrm{M}}_{g-h,S^c\cup\{s\}}$$

where the isomorphism is obtained by attaching the two curves along the last marked points. The problem is while we can inductively understand each irreducible component of boundary of the moduli space, these boundary components intersect. However, the next proposition guarantees that this does not effect the small cohomology groups.

Proposition 2.3. *Let X_i, $i \in I$, denote all the irreducible components of the boundary of $\overline{M}_{g,n}$. The map*

$$H^k(\overline{M}_{g,n},\mathbb{Q}) \to \oplus_{i\in I} H^k(X_i,\mathbb{Q})$$

is injective if $k \leq d(g,n)$.

Sketch. This proposition follows from the fact that the map

$$H^k(\overline{\mathrm{M}}_{g,n},\mathbb{Q}) \to H^k(\delta\mathrm{M}_{g,n},\mathbb{Q})$$

is a morphism of Hodge structures. Since the map is an injection in the claimed range and $H^k(\overline{\mathrm{M}}_{g,n},\mathbb{Q})$ is pure of weight k, the cohomology injects to the weight k part of the cohomology. The proposition follows from a result of Deligne which asserts that if $f: X \to Y$ is a proper, surjective morphism from a smooth variety to a proper variety, then the weight k quotient of $H^k(Y,\mathbb{Q})$ is the image of $H^k(Y,\mathbb{Q})$ in $H^k(X,\mathbb{Q})$. Taking X to be the disjoint union of the irreducible components of the boundary and Y to be the boundary, Deligne's result (at least its modification for orbifolds) implies the proposition. $\square$

Proposition 2.4. *Let $\xi : \overline{M}_{g-1,n+2} \to \overline{M}_{g,n}$ be the morphism that glues the last two marked points. Then the induced map*

$$\xi^* : H^2(\overline{M}_{g-1,n+2}, \mathbb{Q}) \to H^2(\overline{M}_{g,n}, \mathbb{Q})$$

is injective if $g \geq 2$.

Exercise 2.5. Prove the Proposition 2.4 by induction on the number of marked points and the genus. Use Künneth decomposition, Proposition 2.3 and the fact that $H^1(\overline{\mathrm{M}}_{g,n}, \mathbb{Q}) = 0$ for every g and n. There are many ways of proving the last statement. It follows, for example, from the fact that $\overline{\mathrm{M}}_{g,n}$ is simply connected. We will see an elementary proof in the next section.

2.1. **The relations among tautological classes.** In this subsection we indicate how tautological divisor classes pull-back under special morphisms. Let

$$\pi_{n+1} : \overline{\mathrm{M}}_{g,n+1} \to \overline{\mathrm{M}}_{g,n}$$

be the morphism that forgets the $n+1$st marked point.

Exercise 2.6. Prove the following formulae:

(1) $\pi_{n+1}^*(\kappa) = \kappa - \psi_{n+1}$.

(2) $\pi_{n+1}^*(\psi_i) = \psi_i - \delta_{0,\{i,n+1\}}$ for $i \leq n$.

(3) $\pi_{n+1}^*(\delta_{irr}) = \delta_{irr}$.

(4) $\pi_{n+1}^*(\delta_{h,S}) = \delta_{h,S} + \delta_{h,S\cup\{n+1\}}$.

Let

$$\xi : \overline{\mathrm{M}}_{g-1,n\cup\{x,y\}} \to \overline{\mathrm{M}}_{g,n}$$

be the morphism that glues the two points x, y.

Exercise 2.7. Show that ξ pulls back the tautological classes as follows:

(1) $\xi^*(\kappa) = \kappa$.

(2) $\xi^*(\phi_i) = \phi_i$ for $i \leq n$.

(3) $\xi^*(\delta_{irr}) = \delta_{irr} - \psi_x - \psi_y + \sum_{x\in S, y\notin S} \delta_{g,S}$

(4) $\xi^*(\delta_{h,S}) = \begin{cases} \delta_{h,S} & \text{if } \ g = 2h, \quad n = 0 \\ \delta_{h,S} + \delta_{h-1,S\cup\{x,y\}} & \text{otherwise} \end{cases}$

Finally, we need to know the pull-backs of tautological classes by the morphism

$$at_{h,S} : \overline{\mathrm{M}}_{g-h,n-S\cup\{x\}} \to \overline{\mathrm{M}}_{g,n}$$

obtained by attaching a fixed curve of genus h and marking $S \cup \{y\}$ to curves in $\overline{\mathrm{M}}_{g-h,n-S\cup\{x\}}$ by identifying x and y.

Exercise 2.8. Show that the following relations hold:

(1) $at_{h,S}^*(\kappa) = \kappa$.

(2) $at_{h,S}^*(\phi_i) = \begin{cases} \phi_i & \text{if } \ i \in S \\ 0 & \text{otherwise} \end{cases}$

(3) $at_{h,S}^*(\delta_{irr}) = \delta_{irr}$.

(4) If $S = \{1, \ldots, n\}$, then

$$at^*_{h,S}(\delta_{k,T}) = \begin{cases} \delta_{2h-g,S\cup\{x\}} - \psi_x & \text{if } \ k = h, \#T = n, \text{ or } \ k = g-h, \#T = 0 \\ \delta_{k,T} + \delta_{k+h-g,T\cup\{x\}} & \text{otherwise} \end{cases}$$

(5) If $S \neq \{1, \ldots, n\}$, then

$$at^*_{h,S}(\delta_{k,T}) = \begin{cases} -\psi_x & \text{if } \ (k,T) = (h,S) \text{ or } \ (k,T) = (g-h, S^c) \\ \delta_{k,T} & \text{if } \ T \subset S \text{ and } \ (k,T) \neq (h,S) \\ \delta_{k+h-g,(T\setminus S^c)\cup\{x\}} & \text{if } \ S^c \subset T \ \text{ and } \ (k,T) \neq (g-h, S^c) \\ 0 & \text{otherwise} \end{cases}$$

Using the previous three exercises we can obtained the claimed relations in Theorem 2.1. Recall that the Hodge class λ is the first chern class of the Hodge bundle.

Lemma 2.9 (Mumford's relation)**.** *On any* $\overline{\mathrm{M}}_{g,n}$ *there is the following relation*

$$\kappa = 12\lambda - \delta + \psi.$$

Proof. It suffices to prove the formula when $n = 0$. The general case follows by pulling-back via the relations given by the forgetful morphisms. We use the Grothendieck - Riemann - Roch (GRR) formula to see the case $n = 0$. Set

$$\Omega = \Omega^1_{\overline{\mathrm{M}}_{g,1}/\overline{\mathrm{M}}_g}.$$

Recall the GRR formula reads

$$ch(\pi_{1!}F) = \pi_{1*}(ch(F) \cdot Todd(\Omega)).$$

Set $F = \omega_{\overline{\mathrm{M}}_{g,1}/\overline{\mathrm{M}}_g}$. Since $R^1\pi_{1*}$ of the relative dualizing sheaf is trivial, solving for the degree one term of the GRR formula we obtain

$$\lambda = c_1(\pi_{1*}F) = \pi_{1*}(\frac{c_1(\Omega)^2 + c_2(\Omega)}{12} - \frac{c_1(F)c_1(\Omega)}{2} + \frac{c_1(F)}{2})$$

A local calculation shows that

$$c_1(\Omega^1_{\overline{\mathrm{M}}_{g,1}/\overline{\mathrm{M}}_g}) = c_1(\omega^1_{\overline{\mathrm{M}}_{g,1}/\overline{\mathrm{M}}_g}), \quad c_2(\Omega^1_{\overline{\mathrm{M}}_{g,1}/\overline{\mathrm{M}}_g}) = [Sing]$$

where $Sing$ denotes the singular locus. This follows from the exact sequence

$$0 \to \Omega^1_{\overline{\mathrm{M}}_{g,1}/\overline{\mathrm{M}}_g} \to \omega^1_{\overline{\mathrm{M}}_{g,1}/\overline{\mathrm{M}}_g} \to \omega^1_{\overline{\mathrm{M}}_{g,1}/\overline{\mathrm{M}}_g} \otimes \mathcal{O}_{Sing} \to 0.$$

Mumford's formula immediately follows. □

Now all the relations follow when we observe that on $\overline{\mathrm{M}}_2$ we have the relation

$$10\lambda = \delta_0 + 2\delta_1.$$

To prove this relation, for instance, consider the following test families.

(1) To a fixed genus 1 curve attach a fixed point of a genus 1 curve at a variable point.

(2) On a genus 1 curve identify a variable point with a fixed point.

(3) Identify a fixed point of a fixed genus 1 curve with a pencil of plane cubics.

Exercise 2.10. By calculating the intersections of these families with δ_0, δ_1 and λ prove the claimed equality.

Exercise 2.11. Deduce the relations in Theorem 2.1 from the relations in this section.

Remark 2.12. By intersecting with test families it is not hard to show that the relations in Theorem 2.1 are the only relations among tautological divisors.

2.2. **Sketch of the proof of Theorem 2.1.** In this subsection we will sketch the proof of Theorem 2.1. We would like to show that $H^2(\overline{\mathrm{M}}_{g,n}, \mathbb{Q})$ is tautological. Assume that the tautological classes generate the second cohomology of $\overline{\mathrm{M}}_{h,m}$ whenever $h < g$ or $h = g$ and $m < n$. Suppose that the genus is at least 3 for now.

Let $d \in H^2(\overline{\mathrm{M}}_{g,n}, \mathbb{Q})$ be any class. Consider

$$\xi^* d \in H^2(\overline{\mathrm{M}}_{g-1,n\cup\{x,y\}}, \mathbb{Q})$$

where $\xi : \overline{\mathrm{M}}_{g-1,n\cup\{x,y\}} \to \overline{\mathrm{M}}_{g,n}$ is the morphism that identifies the two points x, y. Since by induction $H^2(\overline{\mathrm{M}}_{g-1,n\cup\{x,y\}}, \mathbb{Q})$ is tautological $\xi^* d$ may be expressed as a linear combination of tautological classes. Moreover, since the morphism is symmetric under exchanging x and y, the expressions of divisors involving x and y need to be symmetric. Hence, $\xi^* d$ is a linear combination of κ, ψ_i, $i \leq n$, $\psi_x + \psi_y$, δ_{irr}, $\delta_{h,S}$, $\delta_{h,S\cup\{x,y\}}$ and $\delta_{h,S\cup\{x\}} + \delta_{h,S\cup\{y\}}$.

We can find a tautological class d_t in $H^2(\overline{\mathrm{M}}_{g,n}, \mathbb{Q})$ such that $\xi^*(d - d_t)$ can be expressed only in terms of $\psi_x + \psi_y$, $\delta_{h,S\cup\{x,y\}}$ and $\delta_{h,S\cup\{x\}} + \delta_{h,S\cup\{y\}}$. To conclude that all the coefficients vanish we further pull-back $\xi^*(d - d_t)$ by the morphism

$$ell_{g-2} : \overline{\mathrm{M}}_{g-2,n\cup\{x,y,z\}} \to \overline{\mathrm{M}}_{g-1,n\cup\{x,y\}}$$

obtained by attaching a fixed elliptic tail at the marked point z. We could also pull-back $d - d_t$ to $\overline{\mathrm{M}}_{g-2,n\cup\{x,y,z\}}$ in a different order, first by the map

$$ell_{g-1} : \overline{\mathrm{M}}_{g-1,n\cup\{z\}} \to \overline{\mathrm{M}}_{g,n}$$

that attaches a fixed elliptic curve at the point z, then by the map

$$\xi_{g-2} : \overline{\mathrm{M}}_{g-2,n\cup\{x,y,z\}} \to \overline{\mathrm{M}}_{g-1,n\cup\{z\}}$$

that identifies the points x and y. The classes of these two pull-backs have to coincide. This gives a relation that shows that $\xi^*(d - d_t)$ must be identically zero. Since by Proposition 2.4, the map ξ^* is injective, we conclude that d is tautological.

To conclude the proof then one needs to analyze the cases of genus 1 and 2 in greater detail. This is straightforward but tedious. We leave you to read the details in [AC2].

3. The first, third and fifth cohomology groups of moduli space

The purpose of this section is to sketch an elementary proof of the vanishing of the first, third and fifth cohomology groups of $\overline{\mathrm{M}}_{g,n}$ following Arbarello and Cornalba [AC2].

Theorem 3.1. *$H^k(\overline{M}_{g,n},\mathbb{Q}) = 0$ for $k = 1, 3, 5$.*

The proof proceeds by reducing the general case to checking the vanishing for finitely many $\overline{M}_{g,n}$ with g and n small and carrying out these verifications explicitly. As in the previous section set

$$d(g,n) = \begin{cases} n-4 & \text{if } g = 0 \\ 2g-2 & \text{if } n = 0 \\ 2g-3+n & \text{if } g, n > 0. \end{cases}$$

Recall that $H^k(\overline{M}_{g,n},\mathbb{Q})$ injects into $\oplus_i H^k(X_i,\mathbb{Q})$ where the X_i denote all the irreducible components of the boundary. Like in the previous section we have the following Reduction Lemma.

Lemma 3.2 (Reduction Lemma)**.** *Let k be an odd integer. Suppose that*

$$H^q(\overline{M}_{g,n},\mathbb{Q}) = 0$$

for all odd $q \leq k$, and for all g and n such that $q > d(g,n)$, then

$$H^q(\overline{M}_{g,n},\mathbb{Q}) = 0$$

for all odd $q \leq k$ and all g and n.

In other words, as long as all the odd cohomology for $j < k$ vanishes, to conclude vanishing of the k-th cohomology it suffices to verify it for finitely many special values, namely those values for which $q > d(g,n)$.

Proof. The proof is by induction on k. Suppose $H^q(\overline{M}_{g,n},\mathbb{Q})$ vanishes for all odd $q \leq k$. We can assume $d(g,n) \geq k$. By the previous lemma we conclude that $H^k(\overline{M}_{g,n},\mathbb{Q})$ injects into $H^k(X_i,\mathbb{Q})$. Each X_i is of the form $\overline{M}_{g-1,n+2}$ or a product of $\overline{M}_{a,A}$ and $\overline{M}_{b,B}$ where either $a < g$ or $a = g$ and $|A| < n$. (Similarly for b and B). Using the Künneth formula, we conclude that $H^k(\overline{M}_{g,n},\mathbb{Q})$ injects into a direct sum of $H^k(\overline{M}_{g-1,n+2},\mathbb{Q})$ and $H^l(\overline{M}_{a,A},\mathbb{Q}) \otimes H^m(\overline{M}_{b,B},\mathbb{Q})$ with $l + m = k$. Since either l or m must be odd, all these spaces vanish by the induction hypothesis except possibly for $k = m$ or $k = l$. In this case either the genus is smaller than g or if the genus is equal to g the number of marked points is smaller than n. A double induction concludes the proof. $\square$

Proof of vanishing of the first cohomology. By the Reduction Lemma to prove that the first cohomology groups of $\overline{M}_{g,n}$ vanish we need to check the cases

$$\overline{M}_{0,3}, \overline{M}_{0,4}, \overline{M}_{1,0}.$$

$\overline{M}_{0,3}$ consists of a single point. $\overline{M}_{0,4}$ and $\overline{M}_{1,1}$ are isomorphic to the projective line. The first cohomology of all these spaces vanish. This concludes the proof that $H^1(\overline{M}_{g,n},\mathbb{Q}) = 0$ for all g and n. $\square$

Remark 3.3. $H^1(\overline{M}_{g,n}, \mathbb{Q}) = 0$ also follows from the fact that $\overline{\mathrm{M}}_{g,n}$ is simply connected. However, note that $\mathrm{M}_{g,n}$ is not simply connected. This is one reason why computing the cohomology of the compactified moduli space is simpler. For example, we can identify $M_{0,4}$ with $\mathbb{P}^1$ with three points removed. Fix the three marked points at $0, 1, \infty$. The fourth fixed point is free to vary on the sphere except it cannot be one of the other three marked points. The fundamental group of $\mathbb{P}^1 - \{0, 1, \infty\}$ is the free group on two letters. In particular, the first cohomology group of $\mathbb{P}^1 - \{0, 1, \infty\}$ has rank 2. In contrast we saw above that all odd cohomology groups of $\overline{M}_{0,n}$ vanish.

To emphasize the point, observe that the Euler characteristic of $M_{0,n}$ is given by the formula

$$\chi(M_{0,n}) = (-1)^{(n-3)}(n-3)!.$$

To prove this formula consider the map $M_{0,n} \to M_{0,n-1}$ given by forgetting one of the marked points. This is a fibration with each fiber given by a sphere with $n-1$ points removed. We conclude that the Euler characteristic of $M_{0,n}$ is $(3-n)\chi(M_{0,n-1})$. The result follows by induction. The Euler characteristic of $M_{0,n}$ is negative for even n. At least for those n, the odd cohomology groups cannot vanish.

Proof of the vanishing of third cohomology. To conclude that H^3 vanishes for all $\overline{M}_{g,n}$ we need to check the cases

(1) $g = 0$ and $3 \leq n \leq 6$,
(2) $g = 1$ and $1 \leq n \leq 3$, and
(3) $g = 2$ and $n = 0$ or 1.

We already observed that the odd cohomology of $\overline{\mathrm{M}}_{0,n}$ vanishes. In this range, this is easy to check directly.) $\overline{\mathrm{M}}_{0,3}$ is a point so H^3 clearly vanishes. Both $\overline{\mathrm{M}}_{0,4}$ and $\overline{\mathrm{M}}_{1,1}$ are isomorphic to $\mathbb{P}^1$, hence their third cohomology clearly vanishes.

The moduli spaces $\overline{\mathrm{M}}_{0,5}$ and $\overline{\mathrm{M}}_{1,2}$ both have complex dimension 2 or real dimension 4. By Poincaré duality we conclude that the dimension of H^3 is equal to the dimension of H^1. Since H^1 vanishes we conclude that H^3 vanishes.

To show the vanishing of the third cohomology groups of $\overline{M}_{2,0}$ and $\overline{M}_{2,1}$, we observe that they admit surjective morphisms from $\overline{M}_{0,6}$ and $\overline{M}_{0,7}$, respectively. This suffices to show the vanishing of the third cohomology. Recall that genus 2 curves are all hyperelliptic. They are a double cover of $\mathbb{P}^1$ ramified at six points. Given a Riemann sphere with six marked points take the hyperelliptic curve ramified over these six points. Similarly given a Riemann sphere with seven marked points take the hyperelliptic curve of genus 2 ramified at the first six with one of the points above the seventh point as marked. (Note that since the hyperelliptic involution takes one sheet of the covering to the other, the choice is immaterial.) We conclude that the third cohomology groups of these two spaces vanish.

We are left to consider the case $g = 1$ and $n = 3$. One way to check the vanishing of cohomology groups is to use Euler characteristic considerations. If Y is a quasi-projective variety which has a filtration by closed subvarieties $\overline{Y}_i$

$$Y = \overline{Y}_d \subset \overline{Y}_{d-1} \subset \cdots \subset \overline{Y}_1 \subset \overline{Y}_0$$

so that $Y_i = \overline{Y}_i \backslash \overline{Y}_{i-1}$ is empty or of pure dimension i for every i, then by the exact sequence of cohomology with compact supports the Euler characteristic of Y with cohomology with compact supports is the sum of those of Y_d and $\overline{Y}_{d-1}$. Repeating the process and using Poincaré duality we conclude that the Euler characteristic of $\overline{M}_{g,n}$ is the sum of the Euler characteristics of open strata where we stratify $\overline{M}_{g,n}$ according to graph type.

$\overline{M}_{1,3}$ has complex dimension 3 or real dimension 6. We already know that its first and by Poincaré duality its fifth cohomology groups vanish. The second cohomology group is generated by

$$\kappa, \psi_1, \psi_2, \psi_3, \delta_{irr}, \delta_{0,\{1,2\}}, \delta_{0,\{1,3\}}, \delta_{0,\{2,3\}}, \delta_{0,\{1,2,3\}}.$$

There are 4 independent linear relations among these. Hence the rank of the second (and by Poincaré duality fourth) cohomology groups are 5. If we can show that the Euler characteristic of $\overline{M}_{1,3}$ is twelve, it follows that the third cohomology group has to vanish.

Let us compute that the Euler characteristic of $\overline{M}_{1,3}$ is 12. This is done by splitting $\overline{M}_{1,3}$ to its strata according to topological type. In this computation we need the Euler characteristics of $M_{1,2}$, $M_{1,3}$, $M'_{0,4}$, $M'_{0,5}$, where $M'_{0,4}$ and $M'_{0,5}$ denote the space obtained by taking the quotients of $M_{0,4}$ and $M_{0,5}$ under the operation of interchanging the labeling of two marked points. To calculate the Euler characteristics of the latter two we note that we have morphisms from $M_{0,4}$ and $M_{0,5}$ to these spaces. Both morphisms have degree 2 since the fiber over a point has two points corresponding to the two different ways of ordering the identified marked points. The morphism from $M_{0,4}$ to $M'_{0,4}$ is ramified at one point. If there is only one point over $(0, \infty, 1\ x)$, then there must be an automorphism of the sphere permuting 1 and x and keeping 0 and ∞ fixed. This can only happen if $x = -1$ and the automorphism is multiplication by -1. By the Riemann-Hurwitz formula we conclude that $\chi(M'_{0,4}) = 0$. The map in the case of $M'_{0,5}$ is unramified and therefore $\chi(M'_{0,5}) = 1$.

The Euler characteristics of $M_{1,n}$ can be computed inductively. First, $M_{1,1}$ is the affine line, so its Euler characteristic is 1. It is a fundamental theorem in the theory of elliptic curves that the group of automorphisms fixing a point is a group of order 2 except in two cases. In one case the elliptic curve can be realized as ramified over the points $0, 1, -1, \infty$ of the sphere and it has the extra automorphism coming from rotating the sphere by π along the $0 - \infty$ axis (multiplication by -1). In the other case the elliptic curve can be realized as ramified over the cube roots of unity and ∞. Its automorphism group has order 6 and it can be generated by the usual involution and by multiplication by a cube root of unity (rotation of the underlying sphere around the $0 - \infty$ axis by an angle of $2\pi/3$).

Consider the morphism from $M_{1,2}$ to $M_{1,1}$ given by forgetting the second marked point. The fiber over each point of $M_{1,1}$ is an affine line. Hence, the Euler characteristic of $M_{1,2}$ is 1. Next, consider the morphism from $M_{1,3}$ to $M_{1,2}$. Here we need to break $M_{1,2}$ up to pieces over which the fibers have nice descriptions. First, consider the case where p_2, the second marked point, is a 2-torsion point with respect to p_1. Observe that this space is $M'_{0,4}$ and the fiber of the map over such a point is the sphere with two points removed. Next, there is the case when C is the special curve whose automorphism group has order 6 and p_2 lies above 0. In this case the fiber is also a sphere with two points removed. Finally, there is the case when p_2 is not a 2-torsion point and not the special point considered in the previous case. In this case the fiber is an elliptic curve with two points removed. Adding up the various Euler characteristics we conclude that $\chi(M_{1,3}) = 0$. This information together with an enumeration of the strata of $\overline{M}_{1,3}$ suffices to calculate that the Euler characteristic is 12. Since the Euler characteristic is 12, the third cohomology group must vanish. By the reduction lemma this completes the proof that all the third cohomology groups of $\overline{M}_{g,n}$ vanish. $\square$

The technique for showing that the fifth cohomology groups of $\overline{M}_{g,n}$ vanish is similar. The cases that need to be checked in this case are

(1) $g = 0$ and $n \leq 8$
(2) $g = 1$ and $n \leq 5$
(3) $g = 2$ and $n \leq 3$
(4) $g = 3$ and $n \leq 1$.

We already know the case $g = 0$. The case $g = 1$ and $n \leq 4$ are easy. The remaining cases are more challenging.

Remark 3.4. Arbarello and Cornalba's approach outlined here cannot be applied directly to the odd cohomology groups for $k \geq 11$ since these groups do not always vanish. For example, $H^{11}(\overline{M}_{1,11}, \mathbb{Q})$ does not vanish. Their inductive argument breaks down.

Problem 3.5. Determine $H^7(\overline{\mathrm{M}}_{g,n}, \mathbb{Q})$ and $H^9(\overline{\mathrm{M}}_{g,n}, \mathbb{Q})$.

4. The Picard group of the moduli functor

In this section we will determine the Picard group of the moduli functor following [AC1]. A very good introduction to Picard groups of moduli functors is contained in [Mum].

Let $\overline{\mathcal{M}}_{g,n}$ denote the moduli functor of genus g stable curves with n marked points. Let $(C \to S, \sigma_1, \ldots, \sigma_n)$ denote a family of stable curves of genus g and n marked points parameterized by S. A line bundle on the moduli functor $\overline{\mathcal{M}}_{g,n}$ is an assignment of a line bundle L_C to the base of the family S for every family $C \to S$ and isomorphisms between $L_D \cong \alpha^*(L_C)$ for every fiber diagram

$$\begin{tikzcd} D \arrow[r] \arrow[d] & C \arrow[d] \\ T \arrow[r, "\alpha"'] & S \end{tikzcd}$$

satisfying the cocycle condition.

Similarly let $\mathcal{M}_{g,n}$ denote the moduli functor of genus g smooth curves with n marked points. The Picard group of the functor $\mathcal{M}_{g,n}$ is defined the same way.

The Hodge class λ and the classes of the boundary divisors $\delta_{irr}, \delta_1, \ldots, \delta_{\lfloor g/2 \rfloor}$ are elemements of $Pic(\overline{\mathcal{M}}_g)$. Recall that the Hodge class λ is defined as the class of the determinant of the Hodge bundle which is the push-forward of the relative dualizing sheaf on any family. The class δ_{irr} is the class of the divisor of curves with a non-separating node. The class δ_i is the class of the divisor of curves that contain a node that separates the curve to a subcurve of genus i and genus $g-i$.

Similarly $\lambda, \psi_1, \dots, \psi_n, \delta_{irr}, \delta_{h,S}$ are elements of $Pic(\overline{\mathcal{M}}_{g,n})$. Recall that λ is the Hodge class. The class ψ_i is the class of the cotangent line at the i-th marked point and is formally defined by the pull-back of the relative dualizing sheaf by the section giving the i-th marked point. The classes $\delta_{h,S}$ are the classes of boundary divisors of curves containing a node that separates the curve to a subcurve of genus $1 \leq h \leq \lfloor g/2 \rfloor$ with the marked points p_i for $i \in S \subset \{1, \dots, n\}$ and a residual curve of genus $g-h$ with the remaining marked points. Of course, for the curve to be stable $\#S \geq 2$ if $h = 0$.

Theorem 4.1. *Let $g \geq 3$. The Picard group $Pic(\overline{\mathcal{M}}_g)$ is freely generated by the classes $\lambda, \delta_{irr}, \delta_1, \dots, \delta_{\lfloor g/2 \rfloor}$. The Picard group $Pic(\mathcal{M}_g)$ is freely generated by λ.*

In the rest of the course we will only use Theorem 4.1. However, similar techniques also prove the following more general theorem.

Theorem 4.2. *Let $g \geq 3$. The Picard group $Pic(\overline{\mathcal{M}}_{g,n})$ is freely generated by the classes λ, $\psi_1, \dots, \psi_n$ and the classes of boundary divisors. The Picard group $Pic(\mathcal{M}_{g,n})$ is freely generated by λ and $\psi_1, \dots, \psi_n$.*

Sketch of the proof of Theorem 4.1. We first remark that $Pic(\overline{\mathcal{M}}_g)$ is torsion free and contains $Pic(\overline{\mathrm{M}}_g)$ as a finite index subgroup. To see that $Pic(\overline{\mathcal{M}}_g)$ is torsion free one uses Teichmüller theory. Suppose $Pic(\mathcal{M}_g)$ had a torsion element L of prime order p. Since the p-th power of L is trivial, we can take the p-th root of a nowhere vanishing section to get an unramified $\mathbb{Z}/p\mathbb{Z}$ covering of any family. In particular, we get an unramified covering of Teichmüller space which must split completely. It follows that L has a section over the automorphism free locus. This extends to a holomorphic, nowhere vanishing section of L since the p-th power does. Hence L is trivial. Any class in $Pic(\overline{\mathcal{M}}_g)$ whose restriction to $\mathcal{M}_g$ is trivial is an integral linear combination of the boundary classes. The boundary classes are independent, hence $Pic(\overline{\mathcal{M}}_g)$ is torsion free.

By the calculation of the second homology group of $Pic(\overline{\mathcal{M}}_g)$, we can express any divisor class as a linear combination of

$$\lambda, \delta_{irr}, \delta_1, \dots, \delta_{\lfloor g/2 \rfloor}.$$

The point is to show that it may be expressed as an integral linear combination. The strategy is to construct two different sets of one-parameter families of curves $F_1, \dots, F_{\lfloor g/2 \rfloor + 2}$ and $G_1, \dots, G_{\lfloor g/2 \rfloor + 2}$ such that their intersection matrices with respect to

$$\lambda, \delta_{irr}, \delta_1, \dots, \delta_{\lfloor g/2 \rfloor}$$

are non-singular and have relatively prime determinant. Since the determinant of these matrices times the coefficients of the expressions of any divisor class in terms of

$$\lambda, \delta_{irr}, \delta_1, \dots, \delta_{\lfloor g/2 \rfloor}$$

have to be integral, the theorem follows.

The required families are obtained as follows:

Let K_h be the family consisting of a pencil of hyperplane sections of a K3 surface of degree $2h-2$ to which a fixed curve of genus $g-h$ is attached at a base point of the pencil. It is easy to see that

$$K_h \cdot \delta_{irr} = 18 + 6h, \quad K_h \cdot \delta_h = -1, \quad K_h \cdot \delta_i = 0 \text{ if } i \neq h.$$

The degree of λ on K_h is $h+1$.

Let F_h be the family consisting of three curves C_1, C_2, E of genus $h, g-h-1$ and 1, respectively. Attach C_2 to E at a fixed point, then attach C_1 to E at a fixed point of E_1, but a variable point of E. The degree of λ on this familiy is zero. All the intersections with the boundary divisors vanish unless $i = 1, h$ or $h+1$. The degree of δ_1 on F_h is 1 if $h > 1$, 0 if $g-h-1 > h = 1$ and -1 if $g = 3$ and $g-h-1 = h = 1$. The degree of δ_h on F_h is -1 if $g-h-1 > h = 1$ or if $g-h-1 = h = 1$, 0 if $g-h-1 > h = 1$ and -2 if $g-h-1 = h > 1$.

Let C be the family obtained by attaching a fixed genus $g-3$ curve at fixed 4 points to the base points of a pencil of conics. The degree of λ and δ_i on this family is zero. The degree of δ_{irr} is -1.

Finally let CE be the family obtained by attaching a genus $g-3$ curve at three of the base points of a pencil of conics and a genus one curve at the fourth base point. All the degrees except for the degree of δ_1 vanish on this family. The latter degree is -1.

The theorem follows from these computations. If the genus is $2m+1$, the intersection matrix for the families

$$K_h, C, F_1, \dots, F_m$$

has determinant $(-1)^{m+1}(h+1)$ if $m \geq h \geq 2$. Taking $h = 2$ and $h = 3$ gives two relatively prime determinants. If the genus is $2m+2$, the intersection matrix for the families

$$K_h, C, CE, F_1, \dots, F_m$$

has determinant $(-1)^{m+1}(h+1)$ if $m \geq h \geq 2$. Again taking $h = 2$ and $h = 3$ gives two relatively prime determinants. □

5. The Tautological ring of M_g

In this course we will not have time to discuss the tautological ring. In this section I will give a few references to where you may learn more about it. Many people have worked on it, including Faber, Looijenga, Pandharipande, Graber, Vakil, Getzler, Ionel to mame very few (see, for example, [Fab], [Lo], [FaP1], [FaP2], [GP], [GV1], [V], [GV2]). .

Usually when a moduli space is defined with respect to a universal property, it contains certain tautologically defined Chow classes. The prime example of such Chow classes are the chern classes of the universal tautological and quotient bundles on Grassmannians. The Chow ring of the Grassmannian is generated by these tautological classes.

For the moduli space of curves M_g, it is also possible to define tautological classes. Consider the universal curve

$$\pi_1 : M_{g,1} \to M_g.$$

The first chern class of the relative dualizing sheaf leads to a sequence of classes on M_g. More precisely, let $K = c_1(\omega_{M_{g,1}/M_g})$. Define $\kappa_l = \pi_{1*}K^{l+1}$. These are classes in $A^l(M_g)$. Also on M_g there is a rank g locally free sheaf called the Hodge bundle $\mathbb{E}$. The Hodge bundle is defined by $\mathbb{E} = \pi_{1*}\omega_{M_{g,1}/M_g}$. The chern classes $\lambda_l = c_l(\mathbb{E})$ also define classes in $A^l(M_g)$. Ths subring of the Chow ring generated by these classes is called the tautological ring.

One of the first things to observe is that the cohomology of M_g is not in general tautological. There are many ways to see this. The simplest is to observe that tautological classes are even cohomology classes. Since we have computed the Euler characteristic of the moduli spaces, we can see that the moduli space of curves has odd cohomology classes. There are also explicit constructions of non-tautological classes.

Faber has very detailed conjectures about the structure of the tautological ring. Roughly these conjectures say that the tautological ring of M_g exhibits properties that one would expect the algebraic cohomology ring of a smooth projective variety of dimension $g-2$ to exhibit. For instance that it is Gorenstein with socle in degree $g-2$, satisfies Hard Lefschetz and Hodge Positivity with respect to the class κ_1. Furthermore, Faber conjectures that

$$\kappa_1, \dots \kappa_{\lfloor g/3 \rfloor}$$

generate the ring and gives some explicit relations among these generators. I refer you to the papers cited above for detailed statements and what is known.

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