18.727 Topics in Algebraic Geometry: Algebraic Surfaces
Spring 2008

ALGEBRAIC SURFACES, LECTURE 17

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1. K3 Surfaces (contd.)

Remark. Note that K3 surfaces can only be elliptic over $\mathbb{P}^1$: on a K3 surface, however, one can have many different elliptic fibrations, though not every K3 surface has one.

2. Enriques Surfaces

Recall that such surfaces have $\kappa(X) = 0, K_X \equiv 0, b_2 = 10, b_1 = 0, \chi(\mathcal{O}_X) = 1$. A classical Enriques surface has $p_g = 0, q = 0, \Delta = 0$, while a non-classical Enriques surface has $p_g = 1, q = 1, \Delta = 2$ (which can only happen in characteristic 2). We will discuss only classical Enriques surfaces.

Proposition 1. *For an Enriques surface, $\omega_X \ncong \mathcal{O}_X$, but $\omega_X^2 \cong \mathcal{O}_X$.*

Proof. Since $p_g = 0, \omega_X \ncong \mathcal{O}_X$. By Riemann-Roch, $\chi(\mathcal{O}_X(-K)) = \chi(\mathcal{O}_X) + \frac{1}{2}(-K)(-2K) = \chi(\mathcal{O}_X) = 1$, so $h^0(\mathcal{O}_X(-K)) + h^0(\mathcal{O}_X(2K)) \geq 1$. Since $K_X \not\cong \mathcal{O}_X \implies K_X \not\cong \mathcal{O}_X$, $h^0(\mathcal{O}_X(-K)) = 0$ (since $-K \equiv 0$), and so $h^0(\mathcal{O}_X(2K)) \geq 1$. Since $2K \equiv 0, 2K = 0$, i.e. $\omega_X^2 \cong \mathcal{O}_X$. So the order of K in $\mathrm{Pic}\,(X)$ is 2. Note that $\mathrm{Pic}\,(X) = \mathrm{NS}\,(X)$, because $\mathrm{Pic}^0(X) = 0$ since $q = 0, \Delta = 0$ for classical Enriques surfaces. □

Proposition 2. $\mathrm{Pic}^\tau(X) = \mathbb{Z}/2\mathbb{Z}$*, where the former object is the space of divisors numerically equivalent to zero modulo linear (or algebraic) equivalence, or similarly the torsion part of* NS.

Proof. Let $L \equiv 0$. By Riemann-Roch, $\chi(L) = \chi(\mathcal{O}_X) + \frac{1}{2}L\cdot(L-K) = \chi(\mathcal{O}_X) = 1$ Thus, $h^0(L) \neq 0$ or $h^2(L) = h^0(K-L) \neq 0$. But both L and $K-L$ are $\equiv 0$, so either $L \cong \mathcal{O}_X$ or $\omega \otimes L^{-1} \cong \mathcal{O}_X$, i.e. $L \cong \omega$. □

Proposition 3. *Let X be an Enriques surface. Suppose* $\mathrm{char}(k) \neq 2$. *Then $\exists$ an étale covering X' of degree 2 of X which is a K3 surface, and the fundamental group of X'/X is $\mathbb{Z}/2\mathbb{Z}$.*

Proof. K_X is a 2-torsion divisor class. Let $(f_{ij}) \in Z^1(\{U_i\}, \mathcal{O}_X^*)$ be a cocycle representing K. in $\mathrm{Pic}\,(X) = H^1(X, \mathcal{O}_X^*)$. Since $2K \sim 0$, (f_{ij}^2) is a coboundary, so we can write is as $f_{ij}^2 = \frac{g_i}{g_j}$ on $U_i \cap U_j, g_i \in \Gamma(U_i, \mathcal{O}_X^*)$. Now $\pi : X' \to X$ defined locally by $z_i^2 = g_i$ on U_i given by $\frac{z_i}{z_j} = f_{ij}$. This is étale since $\text{char}(k) \neq 2$. $\omega_{X'} = \pi^*(\omega_X) \implies \kappa(X') = 0$ as well. Since $\chi(\mathcal{O}_{X'}) = 2\chi(\mathcal{O}_X) = 2$, X' is a K3 surface from the classification theorem. □

Remark. Over $\mathbb{C}$, in terms of line bundles, take $X' = \{s \in L \mid \alpha(S^{\otimes 2}) = 1\}$, where $\omega_X = L = \mathcal{O}(K)$ is a line bundle equipped with an isomorphism $\alpha : L^{\otimes 2} \xrightarrow{\sim} \mathcal{O}_X$. The map $L \supset X' \ni s \to (x, x) \in X' \times_X L$ defines a nowhere vanishing section of $\pi^* L$ which is trivial, implying that $\pi^* L = K_{X'}$ is trivial. This implies that $\chi(\mathcal{O}_{X'}) = 2$, and thus X' is K3.

Proposition 4. *Let X' be a K3 surface and i a fixed-point-free involution s.t. it gives rise to an étale connected covering $X' \to X$. If $\text{char}(K) \neq 2$, then X is an Enriques surface.*

Proof. $\omega_{X'} = \pi^*(\omega_X)$, and since $\omega_{X'} \equiv \mathcal{O}_{X'}$, $\omega_X \equiv 0, \kappa(X) = 0$, and $\chi(\mathcal{O}_X) = \frac{1}{2}\chi(\mathcal{O}_{X'}) = 1$. By classification, X is an Enriques surface. □

Thus, Enriques surfaces are quotients of K3 surfaces by fixed-point free involutions.

Example. The smooth complete intersection of 3 quadrics in $\mathbb{P}^5$ is a K3 surface. Let $f_i = Q_i(x_0, x_1, x_2) + Q_i'(x_3, x_4, x_5)$ for $i = 1, 2, 3$, where Q_i, Q_i' are homogeneous quadratic forms; the f_i cut out X', a K3 surface. Now, let $\sigma : \mathbb{P}^5 \to \mathbb{P}^5, \sigma(x_0 : \cdots : x_5) = (x_0 : x_1 : x_2 : -x_3 : -x_4 : -x_5)$ be an involution. Note that $\sigma(X') = X'$. Generically, the 3 conics $Q_i = 0$ in $\mathbb{P}^2$ (respectively the conics $Q_i' = 0$) have no points in common, implying that $\sigma' = \sigma_{X'}$ has no fixed points in X', giving us an Enriques surface as above.

Theorem 1. *Every Enriques surface is elliptic (or quasielliptic).*

Proof. Exercise. □

3. Bielliptic surfaces

This is the fourth class of surfaces with $\kappa(X) = 0 : b_2 = 2, \chi(\mathcal{O}_X) = 0, b_1 = 2, K_X \equiv 0$. There are two cases:

(1) $p_g = 0, q = 1, \Delta = 0$: the classical, bielliptic/hyperelliptic surface.
(2) $p_g = 1, q = 2, \Delta = 2$, which only happens in positive characteristic.

In either case, $b_1 = 2 \implies s = \frac{b_2}{2} = 1 = \dim \text{Alb}\,(X)$, so the Albanese variety is an elliptic curve.

Theorem 2. *The map $f : X \to \text{Alb}\,(X)$ has all fibers either smooth elliptic curves, or all rational curves, each having one singular point which is an ordinary cusp. The latter case happens only in characteristic 2 or 3.*

Proof. Let $B = \mathrm{Alb}\,(X)$, $b \in B$ a closed point, $F = F_b = f^{-1}(b)$. Then $F^2 = 0, F \cdot K = 0 \implies p_a(f) = 1 \implies f : X \to B$ is an elliptic or quasi-elliptic fibration (the latter only in characteristic 2 or 3). All the fibers of f are irreducible (if we had a reducible fiber $F = \sum a_i E_i$, then the classes of F, E_i, and H (the hyperplane section) would give 3 independent classes in $\mathrm{NS}\,(X)$, implying that $b_2 \geq \rho \geq 3$ by the Igusa-Severi inequality, a contradiction). Similarly, one can show that there are no multiple fibers, implying that all fibers are integral. If the general fiber is smooth (or any closed fiber is smooth), then $f^*\omega, \omega \in F^0(B, \omega'_B)$ is a regular 1-form on X, vanishes exactly where f is not smooth, implying that it is a global section of $\Omega^1_{X/k}$ whose zero locus is either empty or of pure codimension 2. A result of Grothendieck shows that the degree of the zero locus is $c_2(\Omega^1_{X/k}) = c_2 = 2 - 2b_1 + b_2 = 0$, implying that $f^*\omega$ is everywhere nonzero and f is smooth. □

Remark. If all fibers of the Albanese map are smooth, call it a hyperelliptic/bielliptic surface. If all fibers of the Albanese map are singular, call it a quasihyperelliptic/quasibielliptic surface.

Next, we find a second elliptic fibration.

Theorem 3. *Let X be as above, $f : X \to B = \mathrm{Alb}\,(X)$ a hyperelliptic or quasihyperelliptic fibration. Then $\exists$ another elliptic fibration $g : X \to \mathbb{P}^1$.*

Proof. (Idea) Find an indecomposable curve C of canonical type s.t. $C \cdot F_t > 0$ for all $t \in B$, where $F_t = f^*(t)$. First note the following.

Definition 1. *Let X be a minimal surface and $D = \sum n_i E_i > 0$ be an effective divisor on X. We say that D is a divisor (or curve)* of canonical type *if $K \cdot E_i = D \cdot E_i = 0$ for all $i = 1, \cdots, r$. If D is also connected, and the g.c.d. of the integers n_i is 1, then we say that D is an indecomposable divisor (or curve) of canonical type.*

Theorem 4. *Let X be a minimal surface with $K^2 = 0$ and $K \cdot C \geq 0$ for all curves C on X. If D is an indecomposable curve of canonical type on X, then $\exists$ an elliptic or quasi-elliptic fibration $f : X \to B$ obtained from the Stein factorization of the morphism $\phi_{|nD|} : X \to \mathbb{P}(H^0(\mathcal{O}_X(nD))^\vee)$ [dual, since the points of x are functionals on $H^0(\mathcal{O}_X(nD))$) for some $n > 0$.*

We will prove this later, and for now, we return to the proof for hyperelliptic surfaces. If we can find such a C of canonical type, then we get an elliptic or quasielliptic fibration $g : X \to B'$ s.t. $(F_t, G_{t'}) > 0$ for all $t \in B, t' \in B'$, where $G'_t - g^{-1}(t')$. If g where quasielliptic, then the general fiber G_t would be a rational curve, implying that $f(G_{t'})$ is a point (since B is an elliptic curve) and $G_{t'} \subset F_t$ for some t, contradicting $(F_t, G_{t'}) > 0$. So g is in fact an elliptic fibration. Similarly, it is not hard to see that the base must be $\mathbb{P}^1$. How do we find C?

Let H be a hyperplane section, F_0 a fiber of f. Let $D = aH + bF_0$ so that $D^2 = 0$, $D \cdot F_t > 0$ (e.g. $b = -H^2, a = 2(H \cdot F_0)$). Then one can prove that, for some $t \in B, D_t = D + F_t - F_0$ has $|D_t| \neq \varnothing$. □

Now we have two different elliptic fibrations "transversal" to each other.

Theorem 5. *Let X, X' be two minimal surfaces with $\kappa(X) \geq 0$ and $\kappa(X') \geq 0$, and let $\phi : X \dashrightarrow X'$ be a birational map. Then ϕ is an isomorphism.*

Proof. Let us show that ϕ is a morphism (the proof for ϕ^{-1} is the same). Resolve ϕ via a sequence of blowups $\pi_i : X_i \to X_{i-1}, X_0 = X$ to obtain a morphism $f : X_n \to X', f = \phi \circ \pi_1 \circ \cdots \circ \pi_n$ with n minimal. If $n = 0$, we are done, so assume $n > 0$. Let E be the exceptional curve of π_n. If $f(E)$ is a point, then we can factor through π_n, contradicting minimality. Thus $f(E)$ is a curve F. Now, $K_{X'} \cdot F \leq K_{X_n} \cdot E = -1$ where the inequality was proved before for blowups. So there is a curve F with $K_{X'} \cdot F < 0$, implying that X' is ruled and contradicting our hypothesis. □

Now, assume that the characteristic of k is neither 2 nor 3, and let X have two fibrations $f : X \to B, g : X \to \mathbb{P}^1$ as above. Let $F_b = f^{-1}(b), F'_c = g^{-1}(c)$. As before, we show that all the fibers of g are irreducible. The reduced fibers are elliptic curves, and the multiple fibers are multiples of elliptic curves. Let $X = \{c \in \mathbb{P}^1 \,|\, F'_c \text{ is a multiple fiber of } g\}$. This is a finite set. If $c \in \mathbb{P}^1 \setminus S$, then $f_c = f|_{F'_c} : F'_c \to B$ is an étale morphism (using Riemann-Hurwitz, and that the genus of F'_c equals the genus of B, 1). f_c induces a homomorphism of algebraic groups $f_c^* : \operatorname{Pic}^0(B) \to \operatorname{Pic}^0(F'_c)$ and $\operatorname{Pic}^0(F'_c)$ acts canonically on $F'_c \cdot L$ as follows. If L is a degree 0 line bundle and $x \in F'_c$, then $(L, x) \mapsto y$, where $L \otimes \mathcal{O}_{F'_c}(x) \cong \mathcal{O}_{F'_c}(y)$. So we get an action of B on F'_c for each $c \in \mathbb{P}^1 \setminus S$. Since $\{f_c^*\}$ is an algebraic family of homomorphisms of algebraic groups, we get an action σ_0 of B on $g^{-1}(\mathbb{P}^1 \setminus S) \subset X$. Thus, every element $b \in B$ defines a rational map $X \dashrightarrow X$, which we can extend to a morphism to get $\sigma : B \times X \to X$.