18.726 Algebraic Geometry
Spring 2009

I particularly recommend Eisenbud-Harris for the material in this section; they give a complete description of the relationship between the two descriptions of a blowup (using charts versus relative Proj).

1 Relative Proj

Let X be a scheme. Let $\mathcal{S} = \oplus_{n=0}^{\infty} \mathcal{S}_n$ be a graded quasicoherent $\mathcal{O}_X$-algebra. For each open affine U in X, we can form the morphism $\operatorname{Proj} \mathcal{S}(U) \to U$; these glue to give a morphism $\mathbf{Proj}\, \mathcal{S} \to X$. The object $\mathbf{Proj}\, \mathcal{S}$ is called the *relative Proj* of $\mathcal{S}$.

We will hereafter assume that $\mathcal{S}_1$ is finitely generated (remember that this is a local notion), and that $\mathcal{S}$ is (locally) generated by $\mathcal{S}_1$ as a $\mathcal{S}_0$-algebra. We also assume that $\mathcal{S}_0$ is a quotient of $\mathcal{O}_X$; Hartshorne assumes that in fact $\mathcal{S}_0 = \mathcal{O}_X$, but I'd rather not do that. Pick an open affine subset U of X and a surjection $\mathcal{O}_U^{\oplus(m+1)} \to \mathcal{S}_1|_U$; we then obtain a surjection $\operatorname{Sym} \mathcal{O}_U^{\oplus(m+1)} \to \mathcal{S}|_U$, where

$$\operatorname{Sym} \mathcal{O}_U^{\oplus(m+1)} = \oplus_{n=0}^{\infty} \operatorname{Sym}^n \mathcal{O}_U^{\oplus(m+1)}.$$

This in turn gives a closed immersion $\mathbf{Proj}\, \mathcal{S} \to \mathbf{Proj} \operatorname{Sym} \mathcal{O}_U^{\oplus(m+1)}$, and the latter is nothing but the projective space $\mathbb{P}_U^n$. Consequently, $\mathbf{Proj}\, \mathcal{S} \to X$ is locally projective.

We say that a morphism $f : Y \to X$ is *projective* if and only if it occurs as a relative Proj. This implies locally projective and is implied by globally projective, but does not coincide with either.

Following EGA, we write $\mathbb{P}(\mathcal{F})$ for $\mathbf{Proj} \operatorname{Sym} \mathcal{F}$ whenever $\mathcal{F}$ is a finitely generated quasicoherent $\mathcal{O}_X$-module.

2 Very ample sheaves

An *immersion* is a morphism $f : Y \to X$ of schemes which on topological spaces is an isomorphism of Y with a locally closed subset (i.e., a closed subset of an open subset) of X, such that for each $y \in Y$ mapping to $x \in X$, the map $f^\sharp : \mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$ is surjective. Any composition of closed immersions and open immersions is an immersion; conversely, if f is an immersion, then it can be written as a closed immersion followed by an open immersion. (Let U be an open subset of X in which Y is closed; then f factors uniquely through the open immersion $U \to X$, and the resulting map $Y \to U$ satisfies Hartshorne's definition of a closed immersion.)

Let $f : Y \to X$ be a morphism. A quasicoherent sheaf $\mathcal{F}$ on Y is *very ample* relative to f if there exists an immersion $Y \to \mathbb{P}(\mathcal{S}_1)$ for some finitely generated quasicoherent $\mathcal{O}_X$-module $\mathcal{S}_1$, under which $\mathcal{F}$ occurs as the pullback of $\mathcal{O}(1)$. Unlike the definition of projectivity, this notion is indeed local on the base.

Lemma. *The morphism $f : Y \to X$ is projective if and only if f is proper and there exists a very ample sheaf relative to f.*

Proof. See Hartshorne, Remark II.5.16.1. $\qquad\square$

The very ample sheaf pulled back from $\mathcal{O}(1)$ can be used to retrieve the morphism to $\mathbb{P}(\mathcal{S}_1)$. Namely, if $\mathcal{S}_1$ is globally finitely generated, any set of generators pull back to sections of the pullback of $\mathcal{O}(1)$, and those define a morphism to projective space. See Hartshorne Theorem II.7.1.

3   Blowups

Here is a neat class of examples of relative Proj. Let $\mathcal{I}$ be a finitely generated ideal sheaf on the scheme X, and put $Y = \mathbb{P}(\mathcal{I})$. We call Y the *blowup of X along $\mathcal{I}$*.

For example, say $X = \operatorname{Spec} k[x,y]$ for k a field, and let $\mathcal{I}$ be the ideal sheaf defined by (x, y). Over $U = D(x) \cup D(y)$, we have an isomorphism $\mathcal{I}|_U \cong \mathcal{O}_U$, so $Y \times_X U \to U$ is an isomorphism. But the fibre over the origin looks like a projective line with homogeneous coordinates x, y.

The blowup defined by $\mathcal{I}$ carries less information than $\mathcal{I}$ itself. For instance, for any locally principal ideal sheaf, the blowup defined by $\mathcal{I}$ is the identity.

Here is a special property of the blowup. For $f : Y \to X$ a morphism and $\mathcal{I}$ an ideal sheaf on X, we may compose the inclusion $\mathcal{I} \to \mathcal{O}_X$ with $f^\sharp : \mathcal{O}_X \to f_*(\mathcal{O}_Y)$ and then perform adjunction to get $f^*\mathcal{I} \to \mathcal{O}_Y$. The image is an ideal sheaf on Y, called the *inverse image ideal sheaf* of $\mathcal{I}$ under f.

Theorem 1. *If $f : Y \to X$ is the blowup defined by the finitely generated ideal sheaf $\mathcal{I}$ on X, then the inverse image ideal sheaf of $\mathcal{I}$ on Y is locally principal.*

Proof. Recall that $Y = \mathbf{Proj}\,\mathcal{S}$ for $\mathcal{S}_n = \operatorname{Sym}^n \mathcal{I}$. In this notation, the inverse image ideal sheaf of $\mathcal{I}$ on Y is simply $\mathcal{O}_Y(1)$, which is locally free. This proves the claim. $\qquad\square$

In fact, f is universal for this property: any morphism $Z \to X$ such that the inverse image ideal sheaf of $\mathcal{I}$ on Z is locally principal factors uniquely through f (Hartshorne, Proposition II.7.14).

More concrete description of the standard example: the blowup of $\operatorname{Spec} k[x,y]$ at (x,y) is covered by the two charts

$$\operatorname{Spec} k[x, y/x] \qquad \operatorname{Spec} k[y, x/y]$$

glued along $\operatorname{Spec} k[x, y, x/y, y/x]$. In fact, any blowup can be described analogously: the blowup of $\operatorname{Spec} A$ along $I = (r_0, \ldots, r_m)$ is covered by $m + 1$ charts, a typical one of which is

$$\operatorname{Spec} A[t_1, \ldots, t_m](t_1 r_0 - r_1, \ldots, t_m r_0 - r_m).$$

The point is that the inverse image ideal sheaf is supposed to become locally principal, so you must force one of the generators $r_0, \ldots, r_m$ to divide into the other ones, and the different

choices for which generator will divide into the others produces the different charts. (Explicit description of the other charts and the glueing is left to the reader.)

A blowup is a special example of a *modification*. The latter is a morphism $f : Y \to X$ of schemes which is proper, surjective, and *birational* (i.e., its restriction to an open dense subset of X is an isomorphism). In the case of a blowup defined by an ideal sheaf, we get an isomorphism over the complement of the closed set defined by the ideal. In fact, under certain circumstances, every modification can be written as a blowup; see Hartshorne Theorem II.7.17. The catch is that the ideal sheaf is not unique; for example, on $\operatorname{Spec} k[x, y]$, the ideals (x, y) and (x^2, xy, y^2) define the same blowup.

4   Chow's lemma

One use of modifications is to turn proper schemes (over some base) into projective schemes.

Theorem 2 (Chow's lemma). *Let $f : X \to S$ be a morphism of finite type. Assume that either S is noetherian, or S is quasicompact and X has finitely many irreducible components. Then there exists a quasiprojective S-scheme X' and a projective surjective morphism $f : X' \to X$ which restricts to an isomorphism over some open $U \subseteq X$ such that $f^{-1}(U) \cong U$ is dense in X'. Moreover, if X is reduced/irreducible/integral, we can ensure that X' is also.*

See EGA 2, Lemme 5.6.1, or for a weaker result, Hartshorne exercise II.4.10.