Lecture 4: Grassmannians, Finite and Affine Morphisms

Remarks on last time

1. Last time, we proved the *Noether normalization lemma*: If A is a finitely generated k-algebra, then, A contains $B \cong k[x_1, \ldots, x_n]$ (free subring) such that A is a finitely generated B-module.

   Question: When is A a finitely generated B-module?
   Answer: If and only if A is a Cohen-Macauley ring. In particular, this doesn't depend on the choice of B (which is *very* not unique...)

2. A remark on the homework problem (Problem 3(e) of Problem Set 2):

   The answer to the optional problem: $|\mathbb{P}^{2n}(\mathbb{F}_q)| = (1 + \ldots + q^{2n}) + q^n$. This is a quadric in $\mathbb{P}^{2n+1}(\mathbb{F}_q)$. The "middle" term q^n also comes up elsewhere and this generalizes to the Weil conjectures.

   Also, the same problem can be used to compute $H^*(Q_{\mathbb{C}})$ (classical topology). This has the same cohomology as projective space for the middle degree. H^* is 1-dimensional in degree $2, 4, \ldots, 4n$ except for H^{2n}, which is 2-dimensional. The fact that the cohomology H^* is the same as for $\mathbb{CP}^n$ except for the middle degree generalizes to the *Lefschetz Hyperplane Theorem*, which will be covered in 18.726.

3. On the isomorphism $X \cong \mathbb{P}^1$ for irreducible degree 2 curves $X \subset \mathbb{P}^2$:

   The degree 2 curve $C = (XY - Z^2)$ in $\mathbb{P}^2$ from last lecture can be covered by two affine open pieces:

   (a) $X \neq 0$: $a = \dfrac{Y}{X}$, $b = \dfrac{Z}{X}$, $(a = b^2) \cong \mathbb{A}^1 = U_1$

   (b) $Y \neq 0$: $a' = \dfrac{X}{Y}$, $b' = \dfrac{Z}{Y}$, $(a' = b'^2) \cong \mathbb{A}^1 = U_2$

   Note that $U_1 \cap U_2 \cong \mathbb{A}^1 \setminus \{0\}$.

   By changing coordinates, we can take the degree 2 curve in $\mathbb{P}^2$ to be $X^2 + Y^2 = Z^2$. Connect points in a quadric to a fixed point. In practice, we can work with the point $(1 : 0 : 1)$. We identify the set of all lines through a given point with $\mathbb{P}^1$. Taking this to affine coordinates, we send $(a, b) \mapsto \dfrac{a-1}{b}$. Writing $a = tb + 1$, we express a and b via t. Then, we get a bijection $\mathbb{P}^1_k \longleftrightarrow X$. This map sends points with rational coordinates to points with rational coordinates. One application is the classification of Pythagorean triples. (Exercise: Work out the details.)

Noetherian topological spaces and irreducible components

Proposition 1. *A Noetherian topological space X is a finite union of its components (i.e. maximal irreducible subsets).*

Remark 1. Here, we can see that the condition that X is Noetherian can be an analogue of compactness.

Lemma 1. *A Noetherian topological space X is a* finite *union of closed irreducible subsets.*

Proof. We are done if X is irreducible. Suppose that X is *not* such a finite union. Write $X = X_1 \cup X_2$, where X_1 and X_2 are proper closed subsets of X. If the claim is false, then one of either X_1 or X_2 is not a union of finitely many irreducibles. Continuing this process, we get a sequence of closed subsets $X \supsetneq X_1 \supsetneq X_2 \supsetneq \cdots$, which contradicts the assumption that X is Noetherian. $\square$

Now we begin the proof of the proposition.

Proof. Write $X = \bigcup_{i=1}^{n} X_i$, where the X_i are closed irreducible subsets of X. Without loss of generality, we can assume that none of the X_i are a subset of another. Then, X_i is not a subset of $\bigcup_{j \neq i} X_j$ (follows from irreducibility). Otherwise, we would have that X_i is a union of proper closed subsets $X_j \cap X_i$. Since every irreducible closed subset $Z \subset X$ is a subset of X_i for some i, the X_i are exactly the components (i.e. maximal irreducible closed subsets) of X. □

Remark 2. A lot of things are *not* Noetherian in the classical topology (e.g. $\mathbb{R}^n$).

Corollary 1. *A radical ideal in a finitely generated ring without nilpotents A is a finite intersection of prime ideals.*

Remark 3. This gives us a correspondence

$$\text{prime ideals of } A \longleftrightarrow \text{irreducible subsets of Spec } A.$$

Proof. Let I be a radical ideal of A. Then, $I = I_Z$ for some closed subset $Z \subset \text{Spec } A$. Since Z is Noetherian, $Z = \bigcup_{i=1}^{n} Z_i$, where the Z_i are irreducible components of Z. Then, $I = \bigcap_{i=1}^{n} I_{Z_i}$. Note that I_{Z_i} is prime since Z_i is irreducible. Thus, I is a finite intersection of prime ideals. □

Claim: Spec A is irreducible if and only if A has no zerodivisors.

Corollary 2. *A closed subset $Z \subset$ Spec A is irreducible if and only if I_Z is prime.*

Now we begin the proof of the claim.

Proof. Let f and g be nonzero elements of $A \subset \text{Fun}_k(\text{Spec } A)$, where $\text{Fun}_k(\text{Spec } A)$ is the set of k-valued functions on Spec A. Suppose that Spec A is irreducible. If $fg = 0$, then $Z_f \cup Z_g = \text{Spec } A$, where Z_f are the zeros of f and Z_g are the zeros of g. If $Z_f, Z_g \subsetneq \text{Spec } A$, then Spec A is reducible. Thus, we must either have $f = 0$ or $g = 0$ and A has no zerodivisors.

Conversely, suppose that Spec A is *not* irreducible. Let $X = \text{Spec } A$. Then, we can write $X = Z_1 \cup Z_2$, where $Z_1, Z_2 \subsetneq X$ are proper closed subsets of X. Since proper closed subsets correspond to nonzero ideals, we can pick nonzero $f \in I_{Z_1}$ and nonzero $g \in I_{Z_2}$. Then, $fg = 0$ and f and g are zerodivisors of A. □

An example of a projective variety (Grassmannians) Last time, we started to discuss some properties of projective varieties and looked at linear subvarieties of $\mathbb{P}^n$. Here is another example of a projective variety.

Example 1. The *Grassmannian* $Gr(k, n)$ is the set of linear subspaces of dimension k in the n-dimensional vector space $K^n := V$. For example, $Gr(1, n) = \mathbb{P}^{n-1}$. Here, we have the "usual" topology and regular functions on $\mathbb{P}^{n-1}$.

In general, the topology and regular functions are characterized as follows:

Let W be a k-dimensional subspace of V with complement U (i.e. $V = W \oplus U$). If $T \in Gr(k, n)$ is transversal to U (i.e. $T \cap U = \{0\}$), then T is the graph of a unique linear map $W \longrightarrow U$. In other words, we have

$$\begin{aligned} \{T \in Gr(k, n) : T \cap U = \{0\}\} &= \text{Hom}_k(W, U) \\ &\cong Mat_{k,n-k}(K) \\ &\cong \mathbb{A}^{k(n-k)}, \end{aligned}$$

where $Mat_{k,n-k}(K)$ is the set of $k \times (n-k)$ matrices with entries in K.

We require that this subset is open and that the isomorphism with $\mathbb{A}^{k(n-k)}$ is an isomorphism of varieties.

Notation: $\mathbb{P}V := \mathbb{P}^n$ is the projectivization of $V = k^n$ (choose a basis for this).

Theorem 1.1. *This defines a projective algebraic variety. The embedding of $Gr(k,n)$ into projective space is defined by $W \mapsto$ the line $\bigwedge^k W \subset \bigwedge^k V$.*

Claim: This map realizes $Gr(k,n)$ as a closed subvariety in $\mathbb{P}\left(\bigwedge^k V\right) = \mathbb{P}^{\binom{n}{k}-1}$.

Example 2. Consider the case $n = 4$ and $k = 2$. These are lines in $\mathbb{P}^3$.

There is a lemma from linear algebra which gives a basic classification of elements of $\bigwedge^2 V$.

Lemma 2. *Take $\omega \in \bigwedge^2 V$. If $\omega = v_1 \wedge v_2$, then $\omega \wedge \omega \in \bigwedge^4 V = 0$. If $\dim V = 4$, then the converse holds.*

Proof. An element ω of $\bigwedge^2 V$ can be thought of as a bilinear skew form (2-form) of the 4-dimensional vector space V^*. Note that $\ker \omega$ is of even dimension. If $\dim \ker \omega = 0$, then $\omega = v_1 \wedge v_2 \wedge v_3 \wedge v_4$ for some basis $\langle v_1, v_2, v_3, v_4 \rangle$ of V. If $\dim \ker \omega = 2$ (pullback from 3-dimensional quotient), then $\omega = v_1 \wedge v_2$ for some v_1, v_2. Finally, $\omega = 0$ if $\dim \ker \omega = 4$, then the form $\omega = 0$.

$\square$

Thus, $Gr(2,4)$ is isomorphic to a quadric in $\mathbb{P}^5$ and $Gr(2,4) \cong Q(\mathbb{P}^5)$, where Q is defined by $\omega \wedge \omega = 0$. (Exercise: Show this is an isomorphism of varieties.) Using some linear algebra, we can show that the quadratic form is not degenerate.

For more details on work above and on Grassmannians in general: See Chapter 6 of *Algebraic Geometry* (1992) by Joe Harris or p. 42 – 44 (in 3rd edition) in Section 1.4.1 ("Closed Subsets of Projective Space") of *Basic Algebraic Geometry 1* by Igor Shafarevich.

Finite and affine morphisms

Definition 1. *A morphism of algebraic varieties $f : X \longrightarrow Y$ is called* affine *if Y has an open cover $Y = \bigcup U_i$ where the U_i are affine open pieces such that the $f^{-1}(U_i) \subset X$ are affine.*

The affine pieces allow us to use commutative algebra. Note that we have an equivalence of categories

$$\{\text{Affine varieties}\} \cong \{\text{Finitely generated } k\text{-algebras with no nilpotents}\},$$

where the second category is the opposite category of the first one.

Definition 2. *The morphism f is* finite *if there is an affine open cover $Y = \bigcup U_i$ such that $f^{-1}(U_i) =$ Spec A and $U_i =$ Spec B with A a finitely generated B-module (see Noether normalization theorem/Noether's lemma).*

This reduces everything to commutative algebra locally on a line.

Lemma 3. *A finite map satisfies the following properties:*

1. *It is closed: $f(Z) \subset Y$ is closed for every closed $Z \subset X$.*

2. *It has finite fibers.*

Corollary 3. *If $B \subset A$ and A is finitely generated over B as a B-module ("A is finite over B"), then Spec $A \longrightarrow$ Spec B has finite* nonempty *fibers.*

Proof. We only need to check that the map Spec $A \longrightarrow$ Spec B is onto. The image is *not* contained in Z_I for all nonzero $I \subset B$ since $B \subset A$. Otherwise, we would have an ideal of B which kills A. Since a finite map is closed, we have that the map is surjective. □

Now we begin the proof of the lemma (use similar ideas as last time) (compare with Lemma 2.4.3 on p. 19 of Kempf).

Proof. Let $f : X \longrightarrow Y$ be a finite map. We can assume X and Y are affine (statement local on line). Since the composition of two finite maps is finite, we can also assume that $Z = X$. Write $X = \text{Spec } A$ and $Y = \text{Spec } B$ and let $I = \text{Ann}_B(A)$. This is a radical ideal since A has no nilpotents. Since I is a radical ideal, it corresponds to the closed subset Z_I of Spec B. Then, we have the surjection $X \twoheadrightarrow Z_I$ and $f(X) \subset Z_I$.

For $x \in Z_I$, we have that $A/\mathfrak{m}_x A \neq 0$ by Nakayama's lemma. Otherwise, there exists $r \equiv 1 \pmod{\mathfrak{m}_x}$ such that $rA = 0$. However, this is not possible since $r \equiv 1 \pmod{\mathfrak{m}_x} \Rightarrow r \notin I$. It follows from Hilbert's Nullstellensatz that $Z_I \subset f(X)$. Since A is a finite B-module, $A/\mathfrak{m}_x A$ is a finite dimensional nonzero k-algebra. This means that there exists a maximal ideal $\mathfrak{m}_x$ such that $\text{Spec } A/\mathfrak{m}_x A = \text{Hom}(A/\mathfrak{m}_x A, k)$ is a *finite* nonempty set (nonempty since quotient ring nonzero). Thus, f has finite nonempty fibers. □

Example 3. *(Examples of affine morphisms)*

1. Let $Z \subset X$ be a closed subvariety. Then, the map $i : Z \hookrightarrow X$ is affine and finite since Spec A/I is a closed subset of Spec A (this is a local question). Any affine open covering of X works.

2. Let Y be any algebraic variety and $X = Y \setminus Z_f$, where $f \in k[X]$. Consider the open embedding $X \hookrightarrow Y$. This map is affine, but usually not finite. Locally, it looks like $\text{Spec } A_{(f)} = A[t]/(1 - tf) \longrightarrow \text{Spec } A$.

Example 4. The morphism $\mathbb{A}^2 \setminus \{0\} \longrightarrow \mathbb{A}^2$ is *not* affine. This is similar to an exercise in the homework (Problem 3 of Problem Set 1). It actually follows from this and the exactness of localization. Let $U \subset \mathbb{A}^2$ be an open neighborhood of 0 such that $U = \mathbb{A}^2 \setminus Z_f$ for some f. Since $k[U] = k[U \setminus \{0\}]$, $U \setminus \{0\}$ is not affine. We also have a short exact sequence

$$0 \longrightarrow k[U \setminus \{0\}] \longrightarrow k[U_1] \oplus k[U_2] \longrightarrow k[U_1 \cap U_2],$$

where $U = U_1 \cup U_2$ ($U_1 = (X \neq 0)$, $U_2 = (Y \neq 0)$). The sequence above is exact because it is obtained from the corresponding sequence in $\mathbb{A}^2$ by localization, which is an exact functor. Thus, there is no affine neighborhood of 0 whose complement is affine.

Preview of next lecture

Lemma 4. *Let $Z_1 \subsetneq Z_2$ be irreducible closed subsets of an algebraic variety X. If $f : X \longrightarrow Y$ is a finite morphism, then $f(Z_1) \subsetneq f(Z_2)$.*

Note that $f(Z_1)$ and $f(Z_2)$ are closed by the previous lemma. We also have that the image of an irreducible set is irreducible. This lemma shows that the images are actually distinct. We will check this result (see Lemma 2.4.4 on p. 19 of Kempf) in the next lecture.

Definition 3. The *dimension* of a Noetherian topological space is the maximal number such that there exists a chain $X \supset Z_n \supsetneq Z_{n-1} \supsetneq Z_{n-2} \supsetneq \cdots \supsetneq Z_0$ of irreducible subsets in X.

For example, the dimension of a point is equal to 0.

Remark 4. The dimension may not necessarily be finite since the Noetherian condition is only for a *given* chain.

Here are some facts about the dimension of a Noetherian topological space:

- $\dim \mathbb{A}^n = n$
- If $X = \bigcup_{i=1}^{n} U_i$, then $\dim X = \max_i \dim U_i$.
- If $f : X \longrightarrow Y$ is a finite and surjective morphism, then $\dim X = \dim Y$.

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