11/19/04

Thue's Theorem. Let a, b, c be nonzero integers. Then the equation $ax^3 + by^3 = c$ has only finitely many solutions.

Diophantine Approximation Thm: Let b be a positive integer that is not a perfect cube, and let $\beta = \sqrt[3]{b}$. Let C be a fixed positive constant. Then there are finitely many pairs of integers (p, q) with $q > 0$ which satisfy

$$\left| \frac{p}{q} - \beta \right| \le \frac{C}{q^3} \qquad *$$

$x^3 - by^3 = c \qquad b, c > 0$

$$\left| \frac{x}{y} - \beta \right| \le \frac{C}{|y|^3} \qquad C = \frac{4|c|}{3\beta^3}.$$

$$x^3 - by^3 = (x - \beta y)(x^2 + \beta y x + \beta^2 y^2)$$

$$p^3 - bq^3 = (p - \beta q)(p^2 + \beta pq + \beta^2 q^2)$$

$$\frac{1}{q^3} \le \left| \frac{p^3 - bq^3}{q^3} \right| = \left| \frac{p}{q} - \beta \right| \left| \left(\frac{p}{q}\right)^2 + \beta \frac{p}{q} + \beta^2 \right|$$

From $(*)$, $\frac{p}{q} \le \beta + \frac{C}{q^3} \le \beta + C$

so $\le \left| \frac{p}{q} - \beta \right| \left| 3\beta^2 + 3\beta C + C^2 \right|.$

$$c' = 3\beta^2 + 3\beta c + c^2$$

$$\left|\frac{p}{q} - \beta\right| \geq \frac{1}{c' q^3}$$

Suppose this held for a smaller exponent:

$$\frac{c}{q^3} \geq \left|\frac{p}{q} - \beta\right| \geq \frac{1}{c' q^{2.9}}$$

$$(cc')^{10} \geq q$$

$$\beta - \frac{c}{q^3} \leq \frac{p}{q} \leq \beta + \frac{c}{q^3}$$

$$\underline{\beta - c \leq \beta q - \frac{c}{q^3} \leq p \leq \beta q + \frac{c}{q^3}} \leq \beta (cc')^{10} + c$$

$$|F(x)| = |x^3 - b| = |(x-\beta)(x^2 + x\beta + \beta^2)|$$

$$\left|F\left(\frac{p}{q}\right)\right| \geq \frac{1}{q^3} \qquad \left|\frac{p}{q} - \beta\right| c'$$

$$F(x) = (x-\beta)^n G(x) \quad \text{integer coefficients.}$$

$$\frac{1}{q^d} \leq \left|F\left(\frac{p}{q}\right)\right| \leq c'' \left|\frac{p}{q} - \beta\right|^n \quad \text{some } c'' \text{ independent of } p, q$$

d = degree of F.

We'd like to have $d \leq 3n$.

but can't since $(x^3-b)^n \mid F(x)$. $\quad d = \deg F(x) \geq 3n$.

New approach.

$F(x, y) \in \mathbb{Z}[x, y]$

F to vanish to higher order at (β, β). and compare upper + lower bounds on

$$F\left(\frac{p_1}{q_1}, \frac{p_2}{q_2}\right) \qquad \frac{p_1}{q_1}, \frac{p_2}{q_2} \text{ both satisfy } *.$$

---

1. Find a good polynomial $F(x, y) \in \mathbb{Z}[x, y]$.

2. Derive an upper bound on $\left|F\left(\frac{p_1}{q_1}, \frac{p_2}{q_2}\right)\right|$ in terms of $\left|\frac{p_1}{q_1} - \beta\right|$ and $\left|\frac{p_2}{q_2} - \beta\right|$.

3. Derive a lower bound on $\left|F\left(\frac{p_1}{q_1}, \frac{p_2}{q_2}\right)\right|$. In particular, show that it is nonzero.

More details. Suppose $\exists$ infinitely many rationals satisfying $*$.

Pick $\frac{p_1}{q_1}$ that satisfy $*$ with q_1 large. Then pick $\frac{p_2}{q_2}$ that satisfies $*$ with q_2 is much larger than q_1.

Consider $\left|F\left(\frac{p_1}{q_1}, \frac{p_2}{q_2}\right)\right|$ and conclude that it's very small.

<u>Step 3.</u> The auxilliary polynomial F does not vanish at $\frac{p_1}{q_1}, \frac{p_2}{q_2}$.

$$\left|F\left(\frac{p_1}{q_1}, \frac{p_2}{q_2}\right)\right| = \left|\frac{\text{nonzero - integer}}{q_1^d q_2^e}\right| \qquad d, e \text{ degrees in } x, y$$

$$\geq \frac{1}{q_1^d q_2^e}$$