18.650. Statistics for Applications
Fall 2016. Problem Set 3

Due Friday, Sep. 30 at 12 noon

Problem 1 Maximum likelihood estimator

Let $X_1, \ldots, X_n$ be n i.i.d. random variables with density f_θ with respect to the Lebesgue measure. For each case below find the MLE of θ.

1. $f_\theta(x) = \theta\tau^\theta x^{-(\theta+1)}\mathbb{1}(x \geq \tau), \theta > 0$, where $\tau > 0$ is a known constant.
2. $f_\theta(x) = \tau\theta^\tau x^{-(\tau+1)}\mathbb{1}(x \geq \theta), \theta > 0$, where $\tau > 0$ is a known constant.
3. $f_\theta(x) = \sqrt{\theta}x^{\sqrt{\theta}-1}\mathbb{1}(0 \leq x \leq 1), \theta > 0$
4. $f_\theta(x) = (x/\theta^2)\exp(-x^2/2\theta^2)\mathbb{1}(x \geq 0), \theta > 0$
5. $f_\theta(x) = \theta\tau x^{\tau-1}\exp(-\theta x^\tau)\mathbb{1}(x \geq 0), \theta > 0$, where $\tau > 0$ is a known constant.

Problem 2 Consistency of the maximum likelihood estimator

Let $X_1, \ldots, X_n$ be n i.i.d. random variables with distribution $\mathcal{N}(\theta, \theta)$, for some unknown $\theta > 0$. Compute the maximum likelihood estimator of θ and show that it is consistent.

Problem 3 Kullback-Leibler divergence

1. Compute the Kullback-Leibler divergence between $P = \mathcal{N}(a, \sigma^2)$ and $Q = \mathcal{N}(b, \sigma^2)$ for $a, b \in \mathbb{R}$, $\sigma^2 > 0$.
2. Compute the Kullback-Leibler divergence between the Bernoulli distributions $P = \mathsf{Ber}(a)$ and $Q = \mathsf{Ber}(b)$ for $a, b \in (0, 1)$.

Problem 4 Total variation distance

1. Compute the total variation between the uniform probability measures on the intervals $[0, s]$ and $[0, t]$, for some given real numbers s, t, with $0 < s \leq t$.
2. If p and q are two numbers in $[0, 1]$, compute the total variation distance between $\mathsf{Ber}(p)$ and $\mathsf{Ber}(q)$.
3. If $X_1, \ldots, X_n$ are n i.i.d. Bernoulli random variables with some parameter $p \in [0, 1]$ and $\bar{X}_n$ is their empirical average, show that the total variation distance between $\mathsf{Ber}(\bar{X}_n)$ and $\mathsf{Ber}(p)$ converges to zero in probability.

4. Show that the Poisson distribution with parameter $1/n$ converges in total variation distance to the Dirac distribution at zero (i.e., the distribution of the random variable that is always equal to zero).