18.440 Midterm 2 Solutions, Fall 2011: 50 minutes, 100 points

1. (20 points) Suppose that a fair die is rolled 72000 times. Each roll turns up a uniformly random member of the set $\{1, 2, 3, 4, 5, 6\}$ and the rolls are independent of each other. For each $j \in \{1, 2, 3, 4, 5, 6\}$ let X_j be the number of times that the die comes up j.

(a) Compute $E[X_3]$ and $\mathrm{Var}[X_3]$. **ANSWER:** Take $n = 72000$, $p = 1/6$. Then $E[X_3] = np = 12000$ and $\mathrm{Var}[X_3] = np(1 - p) = 10000$.

(b) Compute $\mathrm{Var}[X_1 + X_2]$. **ANSWER:** This counts the number of times that either a one or a two comes up. Each die roll has a 1/3 chance of being a 1 or 2. So $\mathrm{Var}[X_1 + X_2] = n(1/3)(2/3) = 16000$.

(c) Use a normal random variable approximation to estimate the probability that $X_3 > 12100$. You may use the function $\Phi(a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-x^2/2} dx$ in your answer. **ANSWER:** 12100 is one standard deviation above the mean. Approximate probability is $1 - \Phi(1)$.

2. (20 points) Suppose that a fair die is rolled just once. Let Y be 1 if the die comes up 3 and zero otherwise. Let Z be 1 if the die comes up 2 and zero otherwise.

(a) Compute the covariance $\mathrm{Cov}(Y, Z)$ and the variances $\mathrm{Var}(Y)$ and $\mathrm{Var}(Z)$. **ANSWER:**
$\mathrm{Cov}(Y, Z) = E[YZ] - E[Y]E[Z] = 0 - 1/36 = -1/36$ and
$\mathrm{Var}(Y) = \mathrm{Var}(Z) = (1/6)(5/6) = 5/36$.

(b) Compute the covariance of $3Y + Z$ and $Y - 3Z$. **ANSWER:**
$\mathrm{Cov}(3Y + Z, Y - 3Z) = 3\mathrm{Var}(Y) - 8\mathrm{Cov}(Y, Z) - 3\mathrm{Var}(Z) = -8\mathrm{Cov}(Y, Z) = 2/9$.

(c) What is the conditional expectation of Y given that $Z = 0$?
ANSWER: 1/5.

3. (20 points) At a certain track competition, ten athletes take turns throwing javelins. Let X_i be the distance that the ith athlete throws the javelin. Suppose that each X_i is an exponential random variable with an expectation of 50 meters and that the X_i are independent of each other.

(a) What is the probability density function for X_1? What is the parameter λ of this exponential random variable? **ANSWER:**
$\lambda = 1/50$ and $f(x) = \lambda e^{-\lambda x}$ if $x > 0$, and 0 otherwise.

(b) Compute the probability that the first athlete throws the javelin more than 50 meters. **ANSWER:** $e^{-50\lambda} = e^{-1}$.

(c) Compute the probability that at least one athlete throws the javelin more than 50 meters. **ANSWER:** $1 - (1 - e^{-1})^{10}$.

(d) Compute $E[\min\{X_1, X_2, \ldots, X_{10}\}]$, i.e., the expectation of the distance that the last place athlete throws the javelin. **ANSWER:** Minimum of ten independent exponentials of rate $\lambda = 1/50$ is exponential of rate $10\lambda = 1/5$. Expectation is $1/(10\lambda) = 5$ meters.

4. (20 points) Let X be a uniformly random variable on $[0, 5]$.

(a) Write the probability density function f_X and the cumulative distribution function F_X. **ANSWER:** $f_X(x) = \begin{cases} 1/5 & 0 \le x \le 5 \\ 0 & \text{otherwise} \end{cases}$.

(b) What is the moment generating function $M_X(t)$? **ANSWER:** $\frac{e^{5t}-1}{5t}$.

(c) Suppose that Y is a random variable for which $M_Y(0) = 1$ and $M'_Y(0) = 1$ and $M''_Y(0) = 2$. What are $E[Y]$, $E[Y^2]$ are $\mathrm{Var}[Y]$? **ANSWER:** $E[Y] = 1$, $E[Y^2] = 2$, and $\mathrm{Var}[Y] = 2 - 1^2 = 1$.

5. (20 points) Suppose that on a certain road, the times at which red cars go by a given spot are given by a Poisson point process with rate $\lambda = 2/\text{hour}$. Suppose that the times at which green cars go by are also given by a Poisson point process of rate $\lambda = 2/\text{hour}$. Similarly, the times at which blue cars go by are given by a Poisson point process of rate $\lambda = 2/\text{hour}$. Suppose that these three Poisson point processes are independent of each other.

(a) Write down the probability density function for the amount of time until the first red car goes by. **ANSWER:** Write $\lambda = 2$. Answer is $\lambda e^{-\lambda x} = 2e^{-2x}$ if $x > 0$, and 0 otherwise.

(c) Compute the expected amount of time until the first car of *any* of the three colors goes by. **ANSWER:** $1/6$ hour, or 10 minutes.

(c) Compute the probability that exactly three red cars go by during the first hour. **ANSWER:** $e^{-\lambda}\lambda^3/3! = 4/(3e^2)$.

(d) Compute the expected amount of time until at least one car of each of the three colors has gone by. (Hint: does this remind you of the radioactive decay problem?) **ANSWER:** $(\frac{1}{6} + \frac{1}{4} + \frac{1}{2}) = \frac{11}{12}$ hours, or 55 minutes.

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