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The spatial median

In one dimension, for any probability distribution function F with a finite first moment, the medians are exactly the values of m for which $\int |x-m|dF(x)$ is minimized, using a definition allowing an interval of medians on which the distribution function F equals 1/2. This characterization allows the definition of median to be extended to more than one dimension. The spatial median was apparently defined and used in the 1930's by Gini and others.

Let $|\cdot|$ be the usual Euclidean norm on $\mathbb{R}^d$, $|x| = (x_1^2 + ... + x_d^2)^{1/2}$. For any $d \geq 1$ and probability measure P on $\mathbb{R}^d$, and any fixed $s_0 \in \mathbb{R}^d$, a *spatial median* of P is defined as any s such that

$$M(s, P, s_0) \ := \ \int |s-x| - |s_0 - x| dP(x)$$

is minimized. Note that if $\int |x|dP(x) < \infty$, a spatial median is any s such that $\int |s-x|dP(x)$ is minimized.

A set C in a Euclidean space $\mathbb{R}^d$ is called *convex* if and only if for any $x, y \in C$ and $0 \leq \lambda \leq 1$ we have $\lambda x + (1-\lambda)y \in C$. A real-valued function f on a convex set C is called *convex* if and only if for all such x, y and λ we have $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$. Here f will be called *strictly* convex if whenever $x \neq y \in C$ and $0 < \lambda < 1$ we have $f(\lambda x + (1-\lambda)y) < \lambda f(x) + (1-\lambda)f(y)$. It's easily seen that a function f on an interval is convex if its second derivative is nonnegative and strictly convex if its second derivative is strictly positive. For example, on $\mathbb{R}^1$, $f(x) = x^2$ is strictly convex and $f(x) = |x|$ is convex but not strictly convex. (On convex functions see e.g. reference RAP, Chapter 6.)

In the next fact the harder part to prove, the uniqueness, is essentially due to J. B. S. Haldane (1948).

Theorem. For any probability measure P on $\mathbb{R}^d$, a spatial median always exists and doesn't depend on s_0. If P is not concentrated in any line, then its spatial median is unique.

Remark. So, the spatial median has a better uniqueness property in higher dimensions than in one dimension. On the other hand in one dimension the median is equivariant under monotone increasing continuous transformations — at least when the median is unique, as for odd sample size, or if we take the interval of all medians when it isn't. In $\mathbb{R}^d$ for $d \geq 2$ the spatial median is equivariant under Euclidean transformations such as rotations, reflections and translations, and under constant multiples, but not under general affine transformations.

Proof. Since

$$m(s, x, s_0) \ := \ |s-x| - |s_0 - x| \ \leq \ |s - s_0|,$$

$g(s) := M(s, P, s_0)$ is always finite. Clearly, it's continuous in s, and goes to ∞ as $s \to \infty$ for fixed s_0. Thus the infimum of $M(s, P, s_0)$ is attained, and a spatial median always exists. Changing s_0 only adds a constant to the integral, so the minimization doesn't depend on s_0.

For any fixed x and s_0, $s \mapsto m(s, x, s_0)$ is a convex function of s. For s in a bounded set, $m(s, x, s_0)$ is bounded uniformly in x. It follows that $M(s, P, s_0)$ is a convex function of s for fixed P and s_0.

Now suppose P is not concentrated in a line. To prove that the spatial median is unique, suppose it is not. Let g have its minimum value at two points $s \neq t$. Since g is convex, it has the same value at all points of the closed line segment joining s to t. On the other hand, $m(\lambda s + (1 - \lambda)t, x, s_0)$ is a convex function of λ, strictly convex if x is not on the line through s and t. Since the set of such x has positive P-probability, $M(\lambda s + (1 - \lambda)t, P, s_0)$ is a strictly convex function of λ, a contradiction, so the minimum is unique. □

Notes. Haldane (1948) proved uniqueness of the spatial median in $\mathbb{R}^k$, $k \geq 2$. (In Haldane's proof, note that $d^2R/dx^2 > 0$ unless $y_r = 0$ for all r, in which case all the observations are on a line.) Haldane gives the proof in detail for a finite sample (empirical measure).

The device of taking $|s - x| - |s_0 - x|$ in place of $|s - x|$, so as to define the spatial median for arbitrary laws (which may not have a first moment), is mentioned for example in Huber (1981, p. 44).

REFERENCES

Haldane, J. B. S. (1948). Note on the median of a multivariate distribution. *Biometrika* **35**, 414-415.

Huber, P. J. (1981). *Robust Statistics.* Wiley, New York. Reprinted, 2004, Wiley-Interscience.

RAP = Dudley, R. M. (1993). *Real Analysis and Probability.* 2d ed., Cambridge University Press, 2002.