18.445 Homework 5, Due May 6th, 2015

Exercise 1. If $X \in L^1(\Omega, \mathscr{F}, \mathbb{P})$, show that the class

$$\{\mathbb{E}[X \mid \mathscr{A}] : \mathscr{A} \text{ sub } \sigma\text{-algebra of } \mathscr{F}\}$$

is Uniformly Integrable.

(1) Show that, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\mathbb{E}[|X| 1_A] \leq \varepsilon, \quad \text{whenever} \quad \mathbb{P}[A] \leq \delta.$$

(2) Show the conclusion.

Exercise 2. Customers arrive in a supermarket as a Poisson process with intensity N. There are N aisles in the supermarket and each customer selects one of them at random, independently of the other customers. Let X_t^N denote the proportion of aisles which remain empty by time t. Show that

$$X_t^N \to e^{-t}, \quad \text{in probability as } N \to \infty.$$

Exercise 3. Let $T_1, T_2, \ldots$ be independent exponential random variables of parameter λ.

(1) For all $n \geq 1$, the sum $S = \sum_{i=1}^n T_i$ has the probability density function

$$f_S(x) = \frac{\lambda^n x^{n-1}}{(n-1)!} e^{-\lambda x}, \quad x > 0.$$

This is called the Gamma(n, λ) distribution.

(2) Let N be an independent geometric random variable with

$$\mathbb{P}[N = n] = \beta(1-\beta)^{n-1}, \quad n = 1, 2, \ldots.$$

Show that $T = \sum_{i=1}^N T_i$ has exponential distribution of parameter $\lambda\beta$.

Exercise 4. Let $(N^i)_{i \geq 1}$ be a family of independent Poisson processes with respective positive intensities $(\lambda_i)_{i \geq 1}$. Then

(1) Show that any two distinct Poisson processes in this family have no points in common.

(2) If $\sum_{i \geq 1} \lambda_i = \lambda < \infty$, then $N_t = \sum_{i \geq 1} N_t^i$ defines the counting process of a Poisson process with intensity λ.

Exercise 5.(Optional, 3 bonus points)

(1) Let $(N_t)_{t\geq 0}$ be a Poisson process with intensity $\lambda > 0$ and let $(X_i)_{i\geq 0}$ be a sequence of i.i.d. random variables, independent of N. Show that if $g(s,x)$ is a function and T_j are jump times of N then

$$\mathbb{E}\left[\exp\left(\theta\sum_{1}^{N_t} g(T_j,X_j)\right)\right] = \exp\left(\lambda\int_0^t ds\mathbb{E}\left[e^{\theta g(s,X)}-1\right]\right).$$

This is called Campbell's Theorem.

(2) Cars arrive at the beginning of a long road in a Poisson stream of intensity λ from time $t = 0$ onwards. A car has a fixed velocity V miles per hour, where $V > 0$ is a random variable. The velocities of cars are i.i.d. and are independent of the arrival process. Cars can overtake each other freely. Show that the number of cars on the first x miles of the road at time t has a Poisson distribution with mean $\lambda\mathbb{E}[\min\{t, x/V\}]$.

Exercise 6. (Optional, 3 bonus points) Customers enter a supermarket as a Poisson process with intensity 2. There are two salesmen near the door who offer passing customers samples of a new product. Each customer takes an exponential time of parameter 1 to think about the new product, and during this time occupies the full attention of one salesman. Having tried the product, customers proceed into the store and leave by another door. When both salesmen are occupied, customers walk straight in. Assuming that both salesmen are free at time 0, find the probability that both are busy at a later time t.