

**18.443 Problem Set 6 Spring 2015**  
**Statistics for Applications**  
**Due Date: 4/3/2015**  
**prior to 3:00pm**

Problems from John A. Rice, Third Edition. [*Chapter.Section.Problem*]  
 For full credit, explain your reasoning and detail any calculations.

1. Problem 9.11.4

Let  $X$  have one of the following distributions:

$X$	$H_0$	$H_1$
$x_1$	.2	.1
$x_2$	.3	.4
$x_3$	.3	.1
$x_4$	.2	.4

(a). Compare the likelihood ratio,  $\Lambda$  for each possible value  $X$  and order the  $x_i$  according to  $\Lambda$ .

$X$	$H_0$	$H_A$	$\Lambda$
$x_1$	.2	.1	2.
$x_2$	.3	.4	3/4
$x_3$	.3	.1	3
$x_4$	.2	.4	2/4

Sorting the  $x_i$  from highest to lowest  $\Lambda$  :  $x_3, x_1, x_2, x_4$

(b). What is the likelihood ratio test of  $H_0$  versus  $H_A$  at level  $\alpha = 0.2$ ?

We rewrite the table, ordering the rows by decreasing  $\Lambda$ . We add a column corresponding to  $P(\Lambda < \Lambda(x_i) | H_0)$  this is the level of the test that rejects  $H_0$  if  $\Lambda < \Lambda(x_i)$

$X$	$H_0$	$H_A$	$\Lambda$	$P(\Lambda(X) < \Lambda(x_i)   H_0)$
$x_3$	.3	.1	3	0.7
$x_1$	.2	.1	2.	0.5
$x_2$	.3	.4	3/4	0.2
$x_4$	.2	.4	2/4	0.

The test that rejects  $H_0$  if  $\Lambda(X) < \Lambda(x_2)$  has level 0.2 and the test that rejects  $H_0$  if  $\Lambda(X) < \Lambda(x_1)$  has level 0.5.

(c). If the prior probabilities are  $P(H_0) = P(H_A)$  then the outcomes favoring  $H_0$  are those for which the posterior probability ratio is greater than 1:

$$\begin{aligned}
 1 &< \frac{P(H_0|X)}{P(H_A|X)} = \frac{P(H_0)P(X|H_0)}{P(H_A)P(X|H_A)} \\
 &= \left(\frac{P(H_0)}{P(H_A)}\right) \cdot \left(\frac{P(X|H_0)}{P(X|H_A)}\right) \\
 &= \left(\frac{P(H_0)}{P(H_A)}\right) \cdot \Lambda(X) \\
 &= \Lambda(X)
 \end{aligned}$$

For equal prior probabilities, the posterior odds is greater than 1 if the likelihood ratio is greater than 1. This is true for outcomes  $x_3$  and  $x_1$ .

(d). What prior probabilities correspond to the decision rules with  $\alpha = 0.2$  and  $\alpha = 0.5$

For  $\alpha = 0.2$ , the rejection region of the test is  $\{x_4\} = \{x : \Lambda < 3/4\}$ .

For the posterior odds to be less than one, the prior odds must be less than  $4/3$ , which corresponds to

$$P(H_0) = 1 - P(H_A) < 4/7.$$

For  $\alpha = 0.5$ , the rejection region of the test is  $\{x_4, x_2\} = \{x : \Lambda < 2\}$ .

For the posterior odds to be less than one, the prior odds must be less than  $1/2$ , which corresponds to

$$P(H_0) = 1 - P(H_A) < 1/3.$$

To include  $\{x_2\}$  in the rejection region it must be that  $P(H_0) > 4/7$ . So for  $\alpha = 0.5$ , the prior probabilities are such that

$$4/7 < P(H_0) = 1 - P(H_A) < 1/3.$$

Note that higher prior probability of  $H_0$  leads to requiring stronger evidence (lower  $\Lambda$ ) to reject  $H_0$ .

## 2. Problem 9.11.5

(a). The significance level of a statistical test is equal to the probability that the null hypothesis is true.

False. In non-Bayesian hypothesis testing, there are no probabilities on the truth of the null (or alternative) hypothesis. The significance level is the conditional probability of rejecting the null hypothesis when the null hypothesis is true. The probability is computed under the conditional distribution of the data under the null hypothesis.

(b). If the significance level of a test is decreased, the power would be expected to increase.

False. If the significance level of a test is decreased, then the test rejects the null hypothesis with lower probability (less frequently) under the null hypothesis. The power of the test is 1 minus the probability of rejecting the null hypothesis evaluated under the conditional probability for a given alternate hypothesis. The power should decrease too since the test rejects less often if it has a lower level .

(c). If a test is rejected at the significance level  $\alpha$ , the probability that the null hypothesis is true equals  $\alpha$ .

False. See (a) – we cannot evaluate the probability of the null hypothesis being true.

(d). The probability that the null hypothesis is falsely rejected is equal to the power of the test.

False. The power of the test is the probability that the null hypothesis is *correctly* rejected.

(e). A type I error occurs when the test statistic falls in the rejection region of the test.

False. When the test statistic falls in the rejection region of the test, the test procedure rejects the null hypothesis. This may lead to a correct decision of the null hypothesis is false, or to an incorrect decision (the type I error) if the null hypothesis is true. We do not know which.

(f). A type II error is more serious than a type I error.

False. Both error types are serious. In Neyman-Pearson testing, we limit the size of the type I error and find the test that minimizes the type II error subject to the constraint. By controlling the size of the type I error, it is given precedence in specifying tests.

(g). The power of a test is determined by the null distribution of the test statistic.

False. The power of a test is the conditional probability of correctly rejecting the null hypothesis when the alternate hypothesis is true. It is a function which is computed for every distribution in the alternate hypothesis.

(h). The likelihood ratio is a random variable.

True. The likelihood ratio is the ratio of the conditional pmf/pdf of the data ( $X$ ) under  $H_0$  and  $H_A$ . It is a function of the data which is a random variable.

3. Problem 9.11.23

Suppose that a 99% confidence interval for the mean  $\mu$  of a normal distribution is found to be

$(-2.0, 3.0)$ . Would a test of  $H_0 : \mu = -3$  versus  $H_A : \mu \neq -3$  be rejected at the 0.01 significance level?

Yes, by the duality of confidence intervals and hypothesis tests (with two-sided alternatives).

The confidence interval contains the true parameter with probability 0.99 (before observing the data). Accepting  $H_0$  if and only if the confidence interval contains  $H_0$  must have level  $1 - 0.99$ .

4. Problem 9.11.26

(a). The generalized likelihood ratio statistic  $\Lambda$  is always less than or equal to 1.

True. 
$$\Lambda = \frac{\max_{\theta \in \Theta_0} Lik(\theta)}{\max_{\theta \in \Theta} Lik(\theta)}$$

where  $\Theta_0 \subset \Theta$ .

(b). If the  $p$ -value is 0.03, the corresponding test will reject at the significance level 0.02.

False. Since the  $p$ -value is greater than 0.02, the data not *surprising enough* to reject the null hypothesis. The  $p$ -value is the lowest significance level of tests that reject the null hypothesis.

(c). If a test rejects at significance level 0.06, then the  $p$ -value is less than or equal to 0.06.

True. The  $p$ -value of the test statistic must be as low or lower than the significance level for the test to reject.

(d). The  $p$ -value of a test is the probability that the null hypothesis is correct.

False. The  $p$ -value is a conditional probability given the null hypothesis is true.

(e). In testing a simple versus simple hypothesis via the likelihood ratio, the  $p$ -value equals the likelihood ratio.

False. The  $p$ -value is a probability computed assuming the null hypothesis is true. For the likelihood ratio test, it is the probability that the likelihood ratio ( $H_0$  to  $H_1$ ) is smaller than the observed value, computed under the conditional probability given  $H_0$ .

(f). If a chi-square test statistic with 4 degrees of freedom has a value of 8.5, the  $p$ -value is less than 0.05

False. Using  $R$ , the  $p$  value of the test statistic is

$$1 - pchisq(8.5, df = 4) = 0.07488723$$

The  $p$ -value is greater than 0.05.

#### 5. Problem 9.11.31

What values of the generalized likelihood ratio  $\Lambda$  are necessary to reject the null hypothesis at the significance level  $\alpha = 0.1$  if the degrees of freedom are 1, 5, 10, 20.

The generalized likelihood ratio statistic  $-LRstat = 2 \log(\Lambda)$  has a (or an approximate) chi-square distribution under the null hypothesis.

The following  $R$  commands compute the table

```
vec.df=c(1,5,10,20)

alpha=0.10

vec.LRstat.criticalValue<-qchisq(1-alpha, df=vec.df)

Lambda= exp(-.5*vec.LRstat.criticalValue)

> data.frame(df=vec.df,
+           LRstat0=vec.LRstat.criticalValue,
+           Lambda=Lambda)
  df LRstat0      Lambda
1  1  2.705543 2.585227e-01
2  5  9.236357 9.870760e-03
3 10 15.987179 3.376200e-04
4 20 28.411981 6.767321e-07
>
```

6. Problem 9.11.39

See *Problem\_9\_39.html*

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