Lecture 22

22.1 One sided hypotheses continued.

It remains to prove the second part of the Theorem from last lecture. Namely, we have to show that for any $\delta \in K_\alpha$

$$\Pi(\delta^*, \theta) \geq \Pi(\delta, \theta) \text{ for } \theta > \theta_0.$$

Let us take $\theta > \theta_0$ and consider two simple hypotheses

$$h_1 : \mathbb{P} = \mathbb{P}_{\theta_0} \text{ and } h_2 : \mathbb{P} = \mathbb{P}_\theta.$$

Let us find the most powerful test with error of type one equal to α. We know that if we can find a threshold b such that

$$\mathbb{P}_{\theta_0}\Big(\frac{f(X|\theta_0)}{f(X|\theta)} < b\Big) = \alpha$$

then the following test will be the most powerful test with error of type 1 equal to α:

$$\delta_\theta = \begin{cases} h_1 : & \frac{f(X|\theta_0)}{f(X|\theta)} \geq b \\ h_2 : & \frac{f(X|\theta_0)}{f(X|\theta)} < b \end{cases}$$

But the monotone likelihood ratio implies that

$$\frac{f(X|\theta_0)}{f(X|\theta)} < b \Leftrightarrow \frac{f(X|\theta)}{f(X|\theta_0)} > \frac{1}{b} \Leftrightarrow V(T, \theta, \theta_0) > \frac{1}{b}$$

and, since now $\theta > \theta_0$, the function $V(T, \theta, \theta_0)$ is strictly increasing in T. Therefore, we can solve this inequality for T and get that $T > c_b$ for some c_b.

This means that the error of type 1 for the test δ_θ can be written as

$$\mathbb{P}_{\theta_0}\Big(\frac{f(X|\theta_0)}{f(X|\theta)} < b\Big) = \mathbb{P}_{\theta_0}(T > c_b).$$

But we chose this error to be equal to $\alpha = \mathbb{P}_{\theta_0}(T > c)$ which means that c_b should be such that

$$\mathbb{P}_{\theta_0}(T > c_b) = \mathbb{P}_{\theta_0}(T > c) \Rightarrow c = c_b.$$

Therefore,we proved that the test

$$\delta_\theta = \begin{cases} h_1 : & T \le c \\ h_2 : & T > c \end{cases}$$

is the most powerful test with error of type 1 equal to α.

But this test δ_θ is exactly the same as δ^* and it does not depend on θ. This means that deciding between two simple hypotheses θ_0 vs. θ one should always use the same most powerful decision rule δ^*. But this means that δ^* is uniformly most powerful test - what we wanted to prove. Notice that MLR played a key role here because thanks to MLR the decision rule δ_θ was independent of θ. If δ_θ was different for different θ this would mean that there is no UMP for composite hypotheses because it would be advantageous to use different decision rules for different θ.

$\square$

Example. Let us consider a family of normal distributions $N(\mu, 1)$ with unknown mean μ as a parameter. Given some μ_0 consider one sided hypotheses

$$H_1 : \mu \le \mu_0 \text{ and } H_2 : \mu > \mu_0.$$

As we have shown before the normal family $N(\mu, 1)$ has monotone likelihood ratio with $T(X) = \sum_{i=1}^n X_i$. Therefore, the uniformly most powerful test with level of significance α will be as follows:

$$\delta^* = \begin{cases} H_1 : & \sum_{i=1}^n X_i \le c \\ H_2 : & \sum_{i=1}^n X_i > c. \end{cases}$$

The threshold c is determined by

$$\alpha = \mathbb{P}_{\mu_0}(T > c) = \mathbb{P}_{\mu_0}(\sum X_i > c).$$

If the sample comes from $N(\mu_0, 1)$ then T has distribution $N(n\mu_0, n)$ and

$$Y = \frac{1}{\sqrt{n}} \sum_{i=1}^n (X_i - \mu_0) \sim N(0, 1)$$

is standard normal. Therefore,

$$\alpha = \mathbb{P}_{\mu_0}(\sum_{i=1}^n X_i > c) = \mathbb{P}_{\mu_0}\Big(Y = \frac{1}{\sqrt{n}} \sum_{i=1}^n (X_i - \mu_0) > \frac{c - n\mu_0}{\sqrt{n}}\Big).$$

Therefore, if using the table of standard normal distribution we find c_α such that $\mathbb{P}(Y > c_\alpha) = \alpha$ then

$$\frac{c - n\mu_0}{\sqrt{n}} = c_\alpha \text{ or } c = \mu_0 n + \sqrt{n} c_\alpha.$$

Example. Let us now consider a family of normal distributions $N(0, \sigma^2)$ with variance σ^2 as unknown parameter. Given σ_0^2 we consider one sided hypotheses

$$H_1 : \sigma^2 \le \sigma_0^2 \text{ and } H_2 : \sigma^2 > \sigma_0^2.$$

Let us first check if MLR holds in this case. The likelihood ratio is

$$\begin{aligned} \frac{f(X|\sigma_2^2)}{f(X|\sigma_1^2)} &= \frac{1}{(\sqrt{2\pi}\sigma_2)^n} e^{-\frac{1}{2\sigma_2^2}\sum_{i=1}^n X_i^2} \Big/ \frac{1}{(\sqrt{2\pi}\sigma_1)^n} e^{-\frac{1}{2\sigma_1^2}\sum_{i=1}^n X_i^2} \\ &= \Big(\frac{\sigma_1}{\sigma_2}\Big)^n e^{\left(\frac{1}{2\sigma_1^2} - \frac{1}{2\sigma_2^2}\right)\sum X_i^2} = \Big(\frac{\sigma_1}{\sigma_2}\Big)^n e^{\left(\frac{1}{2\sigma_1^2} - \frac{1}{2\sigma_2^2}\right)T}, \end{aligned}$$

where $T = \sum_{i=1}^n X_i^2$. When $\sigma_2^2 > \sigma_1^2$ the likelihood ratio is increasing in T and, therefore, MLR holds. By the above Theorem, the UMP test exists and is given by

$$\delta^* = \begin{cases} H_1 : & T = \sum_{i=1}^n X_i^2 \le c \\ H_2 : & T = \sum_{i=1}^n X_i^2 > c \end{cases}$$

where the threshold c is determined by

$$\alpha = \mathbb{P}_{\sigma_0^2}\Big(\sum_{i=1}^n X_i^2 > c\Big) = \mathbb{P}_{\sigma_0^2}\Big(\sum_{i=1}^n \Big(\frac{X_i}{\sigma_0}\Big)^2 > \frac{c}{\sigma_0^2}\Big).$$

When $X_i \sim N(o, \sigma_0^2)$, $X_i/\sigma_0 \sim N(0,1)$ are standard normal and, therefore,

$$\sum_{i=1}^n \Big(\frac{X_i}{\sigma_0}\Big)^2 \sim \chi_n^2$$

has χ_n^2 distribution with n degrees of freedom. If we find c_α such that $\chi_n^2(c_\alpha, \infty) = \alpha$ then $c = c_\alpha \sigma_0^2$.