

Lecture 17

17.1 Confidence intervals for parameters of normal distribution.

We know by LLN that sample mean and sample variance converge to mean α and variance σ^2 :

$$\bar{X} \rightarrow \alpha, \bar{X}^2 - (\bar{X})^2 \rightarrow \sigma^2.$$

In other words, these estimates are consistent. In this lecture we will try to describe precisely, in some sense, how close sample mean and sample variance are to these unknown parameters that they estimate.

Let us start by giving a definition of a confidence interval in our usual setting when we observe a sample X_1, \dots, X_n with distribution \mathbb{P}_{θ_0} from a parametric family $\{\mathbb{P}_{\theta} : \theta \in \Theta\}$, and θ_0 is unknown.

Definition: Given a parameter $\alpha \in [0, 1]$, which we will call confidence level, if there are two statistics

$$S_1 = S_1(X_1, \dots, X_n) \text{ and } S_2 = S_2(X_1, \dots, X_n)$$

such that the probability

$$\mathbb{P}_{\theta_0}(S_1 \leq \theta_0 \leq S_2) = \alpha, \text{ (or } \geq \alpha)$$

then we call the interval $[S_1, S_2]$ a *confidence interval* for the unknown parameter θ_0 with the confidence level α .

This definition means that we can guarantee with probability/confidence α that our unknown parameter lies within the interval $[S_1, S_2]$. We will now show how in the case of normal distribution $N(\alpha, \sigma^2)$ we can use the estimates (sample mean and sample variance) to construct the confidence intervals for unknown α_0 and σ_0^2 .

Let us recall from the lecture before last that we proved that when

$$X_1, \dots, X_n \text{ are i.d.d. with distribution } \sim N(\alpha_0, \sigma_0^2)$$

then

$$A = \frac{\sqrt{n}(\bar{X} - \alpha_0)}{\sigma_0} \sim N(0, 1) \text{ and } B = \frac{n(\bar{X}^2 - (\bar{X})^2)}{\sigma_0^2} \sim \chi_{n-1}^2$$

and the random variables A and B are independent. If we recall the definition of χ^2 distribution, this mean that we can represent A and B as

$$A = Y_1 \text{ and } B = Y_2^2 + \dots + Y_n^2$$

for some Y_1, \dots, Y_n i.d.d. standard normal.

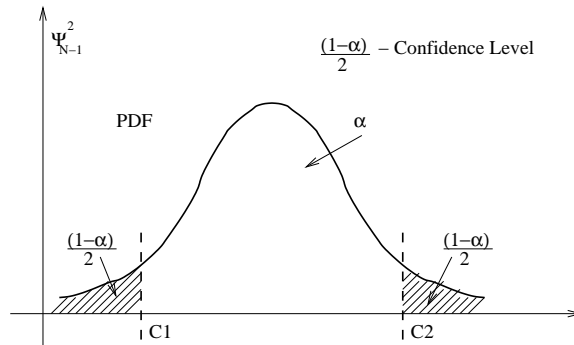


Figure 17.1: P.d.f. of χ_{n-1}^2 distribution and α confidence interval.

First, if we look at the p.d.f. of χ_{n-1}^2 distribution (see figure 17.1) and choose the constants c_1 and c_2 so that the area in each tail is $(1 - \alpha)/2$, since the area represents the probability of the corresponding interval, we get that,

$$\mathbb{P}(B \leq c_1) = \frac{1 - \alpha}{2} \text{ and } \mathbb{P}(B \geq c_2) = \frac{1 - \alpha}{2}.$$

The remaining probability is

$$\mathbb{P}(c_1 \leq B \leq c_2) = \alpha,$$

which means that we can guarantee with probability α that

$$c_1 \leq \frac{n(\bar{X}^2 - (\bar{X})^2)}{\sigma_0^2} \leq c_2.$$

Solving this for σ_0^2 gives

$$\frac{n(\bar{X}^2 - (\bar{X})^2)}{c_2} \leq \sigma_0^2 \leq \frac{n(\bar{X}^2 - (\bar{X})^2)}{c_1}.$$

This precisely means that the interval

$$\left[\frac{n(\bar{X}^2 - (\bar{X})^2)}{c_2}, \frac{n(\bar{X}^2 - (\bar{X})^2)}{c_1} \right]$$

is the α confidence interval for the unknown variance σ_0^2 .

Next, let us construct the confidence interval for the mean α_0 . Consider the following expression,

$$\frac{A}{\sqrt{\frac{1}{n-1}B}} = \frac{Y_1}{\sqrt{\frac{1}{n-1}(Y_2^2 + \dots + Y_n^2)}} \sim t_{n-1}$$

which, by definition, has t -distribution with $n - 1$ degrees of freedom. On the other hand,

$$\frac{A}{\sqrt{\frac{1}{n-1}B}} = \frac{\sqrt{n} \frac{(\bar{X} - \alpha_0)}{\sigma_0}}{\sqrt{\frac{1}{n-1} \frac{n(\bar{X}^2 - (\bar{X})^2)}{\sigma_0^2}}} = \frac{\bar{X} - \alpha_0}{\sqrt{\frac{1}{n-1}(\bar{X}^2 - (\bar{X})^2)}}.$$

If we now look at the p.d.f. of t_{n-1} distribution (see figure 17.2) and choose the constants $-c$ and c so that the area in each tail is $(1 - \alpha)/2$, (the constant is the same on each side because the distribution is symmetric) we get that with probability α ,

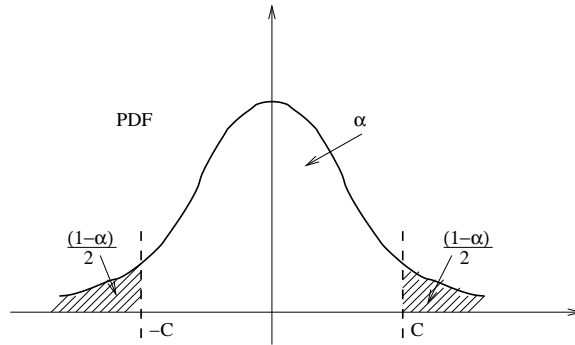


Figure 17.2: t_{n-1} distribution.

$$-c \leq \frac{\bar{X} - \alpha_0}{\sqrt{\frac{1}{n-1}(\bar{X}^2 - (\bar{X})^2)}} \leq c$$

and solving this for α_0 , we get the confidence interval

$$\bar{X} - c\sqrt{\frac{1}{n-1}(\bar{X}^2 - (\bar{X})^2)} \leq \alpha_0 \leq \bar{X} + c\sqrt{\frac{1}{n-1}(\bar{X}^2 - (\bar{X})^2)}.$$

Example. (Textbook, Section 7.5, p. 411). Consider a normal sample of size $n = 10$:

$$0.86, 1.53, 1.57, 1.81, 0.99, 1.09, 1.29, 1.78, 1.29, 1.58.$$

We compute the estimates

$$\bar{X} = 1.379 \text{ and } \bar{X}^2 - (\bar{X})^2 = 0.0966.$$

Choose confidence level $\alpha = 95\% = 0.95$.

We have to find c_1, c_2 and c as explained above. Using the table for t_9 distribution on page 776, we need to find c such that

$$t_9(-\infty, c) = 0.975$$

which gives us $c = 2.262$. To find c_1 and c_2 we can use χ_9^2 table on page 774,

$$\chi_9^2([0, c_1]) = 0.025 \Rightarrow c_1 = 2.7$$

$$\chi_9^2([0, c_2]) = 0.975 \Rightarrow c_2 = 19.02.$$

Plugging these into the formulas above, with probability 95% we can guarantee that

$$\begin{aligned} \bar{X} - c\sqrt{\frac{1}{9}(\bar{X}^2 - (\bar{X})^2)} \leq \alpha_0 \leq \bar{X} + c\sqrt{\frac{1}{9}(\bar{X}^2 - (\bar{X})^2)} \\ 0.6377 \leq \alpha_0 \leq 2.1203 \end{aligned}$$

and with probability 95% we can guarantee that

$$\frac{n(\bar{X}^2 - (\bar{X})^2)}{c_2} \leq \sigma_0^2 \leq \frac{n(\bar{X}^2 - (\bar{X})^2)}{c_1}$$

or

$$0.0508 \leq \sigma_0^2 \leq 0.3579.$$

These confidence intervals may not look impressive but the sample size is very small here, $n = 10$.