Lecture 13

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1 Outline

Last time, we proved the Brunn-Minkowski inequality for boxes. Today we'll go over the general version of the Brunn-Minkowski inequality and then move on to applications, including the Isoperimetric inequality and Grunbaum's theorem.

2 The Brunn-Minkowski inequality

Theorem 1 *Let $A, B \subseteq \mathbb{R}^n$ be compact measurable sets. Then*

$$(\mathrm{Vol}(A \oplus B))^{1/n} \geq (\mathrm{Vol}(A))^{1/n} + (\mathrm{Vol}(B))^{1/n}. \tag{1}$$

The equality holds when A is a translation of a dilation of B (up to zero-measure sets).

Proof An equivalent version of Brunn-Minkowski inquality is given by

$$\Big(\mathrm{Vol}(\lambda A \oplus (1-\lambda)B)\Big)^{1/n} \geq \lambda(\mathrm{Vol}(A))^{1/n} + (1-\lambda)(\mathrm{Vol}(B))^{1/n}, \quad \forall \lambda \in [0,1]. \tag{2}$$

The equivalence of (1) and (2) follows from the fact that $\mathrm{Vol}(\lambda A) = \lambda^n \mathrm{Vol}(A)$:

$$\begin{aligned} \Big(\mathrm{Vol}(\lambda A \oplus (1-\lambda)B)\Big)^{1/n} &\geq \big(\mathrm{Vol}(\lambda A)\big)^{1/n} + \big(\mathrm{Vol}((1-\lambda)B)\big)^{1/n} \\ &= \big(\lambda^n \mathrm{Vol}(A)\big)^{1/n} + \big((1-\lambda)^n \mathrm{Vol}(B)\big)^{1/n} \\ &= \lambda\big(\mathrm{Vol}(A)\big)^{1/n} + (1-\lambda)\big(\mathrm{Vol}(B)\big)^{1/n}. \end{aligned} \tag{3}$$

The inequality (2) implies that the n^{th} root of the volume function is concave with respect to the Minkowski sum.

Here, we sketch the proof for Theorem 1 by proving (1) for any set constructed from a finite collection of boxes. The proof can be generalized to any measurable set by approximating the set with a sequence of finite collections of boxes and taking the limit. We omit the analysis details here.

Let A and B be finite collections of boxes in $\mathbb{R}^n$. We prove (1) by induction on the number of boxes in $A \cup B$. Define the following subsets of $\mathbb{R}^n$:

$$\begin{aligned} A^+ = A \cap \{x \in \mathbb{R}^n | x_n \geq 0\} \ , \ A^- &= A \cap \{x \in \mathbb{R}^n | x_n \leq 0\}, \\ B^+ = B \cap \{x \in \mathbb{R}^n | x_n \geq 0\} \ , \ B^- &= B \cap \{x \in \mathbb{R}^n | x_n \leq 0\}. \end{aligned} \tag{4}$$

Translate A and B such that the following conditions hold:

1. A has some pair of boxes separated by the hyperplane $\{x \in \mathbb{R}^n | x_1 = 0\}$. i.e. there exists a box that lies completely in the halfspace $\{x \in \mathbb{R}^n | x_1 \geq 0\}$ and there is some other box that lies in its complement half-space (see figure 1). (If there's no such box in that direction we can change coordinates.)

2. It holds that
$$\frac{\mathrm{Vol}(A^+)}{\mathrm{Vol}(A)} = \frac{\mathrm{Vol}(B^+)}{\mathrm{Vol}(B)}. \tag{5}$$

Note that translation of A or B just translates $A \oplus B$, so any statement about the translated sets holds for the original ones.

Since A^+ and A^- are strict subsets of A, we know that $A^+ \cup B^+$ and $A^- \cup B^-$ have fewer boxes than $A \cup B$. Therefore, (1) is true for them by the induction hypothesis. Moreover, $A^+ \oplus B^+$ and $A^- \oplus B^-$ are disjoint because they differ in sign of the x_1 coordinate. Hence, we have

$$\begin{aligned}
\mathrm{Vol}(A \oplus B) &\geq \mathrm{Vol}(A^+ \oplus B^+) + \mathrm{Vol}(A^- \oplus B^-) \\
&\geq \left(\mathrm{Vol}(A^+)^{1/n} + \mathrm{Vol}(B^+)^{1/n}\right)^n + \left(\mathrm{Vol}(A^-)^{1/n} + \mathrm{Vol}(B^-)^{1/n}\right)^n \\
&= \mathrm{Vol}(A^+)\left(1 + \left(\frac{\mathrm{Vol}(B^+)}{\mathrm{Vol}(A^+)}\right)^{1/n}\right)^n + \mathrm{Vol}(A^-)\left(1 + \left(\frac{\mathrm{Vol}(B^-)}{\mathrm{Vol}(A^-)}\right)^{1/n}\right)^n \\
&= \left(\mathrm{Vol}(A^+) + \mathrm{Vol}(A^-)\right)\left(1 + \left(\frac{\mathrm{Vol}(B)}{\mathrm{Vol}(A)}\right)^{1/n}\right)^n \\
&= \left(\mathrm{Vol}(A)^{1/n} + \mathrm{Vol}(B)^{1/n}\right)^n, \qquad (6)
\end{aligned}$$

where the second inequality follows from the induction hypothesis, and the second equality is implied by (5).
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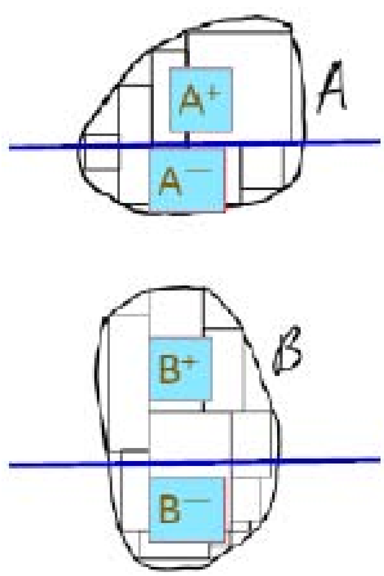

Figure 1: A^+ and B^+ as defined in the proof of Theorem 1.

3 Applications of Brunn-Minkowski Inequality

In this section, we demonstrate the power of Brunn-Minkowski inequality by using it to prove some important theorems in convex geometry.

3.1 Volumes of Parallel Slices

Let $K \in \mathbb{R}^n$ be a convex body. A *parallel slice*, denoted by K_t, is defined as an intersection of the body with a hyperplane, i.e.

$$K_t = K \cap \{x \in \mathbb{R}^n | x_1 = t\}. \qquad (7)$$

Define the volume of the parallel slice K_t, denoted by $v_K(t)$, to be its $(n-1)$-dimensional volume.

$$v_K(t) = \mathrm{Vol}_{n-1}(K_t). \tag{8}$$

We are interested in the behavior of the function $v_K(t)$, and in particular, in whether it is concave.

Consider the Euclidean ball in $\mathbb{R}^n$. The following plots of $v_K(t)$ for different n suggest that except for $n = 2$, the function $v_K(t)$ is not concave in t.

As another example, consider a circular cone in $\mathbb{R}^3$. The volume of a parallel slice is proportional to t^2, so $v_K(t)$ is not concave. More generally, $v_K(t)$ is proportional to t^{n-1} for a circular cone in $\mathbb{R}^n$. This suggests that the $(n-1)^{th}$ root of v_K is a concave function. This guess is verified by Brunn's theorem.

Theorem 2 *(Brunn's Theorem) Let K be a convex body, and let $v_K(t)$ be defined as in (8). Then the function $v_K(t)^{\frac{1}{n-1}}$ is concave.*

Proof Let $s, r, t \in \mathbb{R}$ with $s = (1-\lambda)r + \lambda t$ for some $\lambda \in [0,1]$. Define the $(n-1)$-dimensional slices K_r, K_s, K_t as in (7). First, we claim that

$$(1-\lambda)A_r \oplus \lambda A_t \subseteq A_s. \tag{9}$$

We show this by proving that for any $x \in A_r, y \in A_t$, we have $z = (1-\lambda)x \oplus \lambda y \in A_s$, as follows. Connect the points (r, x) and (t, y) with a straight line (see figure 2). By convexity of K, the line lies completely in the body. In particular, the point (s, z), which is a convex combination of (r, x) and (t, y), lies in A_s. Therefore, $z \in A_s$ and the claim in (9) is true. Now, by applying the version of Brunn-Minkowski inequality in (2), we have

$$\begin{aligned} \mathrm{Vol}(A_s)^{\frac{1}{n-1}} &\geq (1-\lambda)\mathrm{Vol}(A_r)^{\frac{1}{n-1}} + \lambda \mathrm{Vol}(A_t)^{\frac{1}{n-1}} \\ \Rightarrow v_K(s)^{\frac{1}{n-1}} &\geq (1-\lambda)v_K(r)^{\frac{1}{n-1}} + \lambda v_K(t)^{\frac{1}{n-1}} \end{aligned} \tag{10}$$

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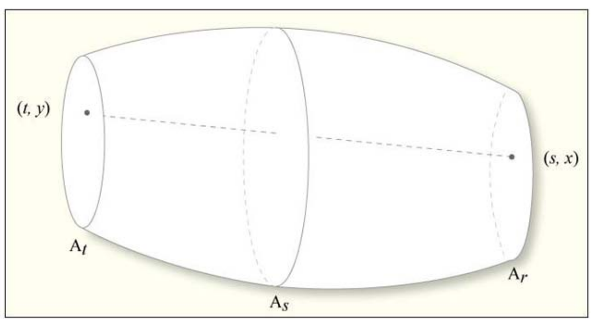

Figure by MIT OpenCourseWare.

Figure 2: n-dimensional convex body K in Theorem 2.

3.2 Isoperimetric Inequality

A few lectures ago, we asked the question of finding the body of a given volume with the smallest surface area. The answer, namely the Euclidean ball, is a direct consequence of the Isoperimetric inequality. Before stating the theorem, let us define the surface area of a body using the Minkowski sum.

Definition 3 *Let K be a body. The surface area of K is defined as the differential rate of volume increase as we add a small Euclidean ball to the body, i.e.,*

$$S(K) = \mathrm{Vol}(\partial K) = \lim_{e \to 0} \frac{\mathrm{Vol}(K \oplus \epsilon B_2^n) - \mathrm{Vol}(K)}{\epsilon}. \tag{11}$$

Now we state the theorem:

Theorem 4 *(Isoperimetric inequality) For any convex body K, with n-dimensional volume $V(K)$ and surface area $S(K)$,*

$$\left(\frac{V(K)}{V(B_2^n)}\right)^{1/n} \leq \left(\frac{S(K)}{S(B_2^n)}\right)^{\frac{1}{n-1}} \tag{12}$$

Proof By applying the Brunn-Minkowski inequality, we have the following:

$$\begin{aligned} V(K \oplus \epsilon B_2^n) &\geq \left[V(K)^{1/n} + \epsilon V(B_2^n)^{1/n}\right]^n \\ &= V(K)\left[1 + \epsilon\left(\frac{V(B_2^n)}{V(K)}\right)^{1/n}\right] \\ &\geq V(K)\left[1 + n\epsilon\left(\frac{V(B_2^n)}{V(K)}\right)\right] \end{aligned} \tag{13}$$

where the second inequality is obtained by keeping the first two terms of the Taylor expansion of $(1+x)^n$.

Now, the definition of surface area in (11) implies:

$$\begin{aligned} S(K) = V(\partial K) &\geq \frac{V(K) + n\epsilon V(K)\left(\frac{V(B_2^n)}{V(K)}\right)^{1/n} - V(K)}{\epsilon} \\ &= nV(K)\left(\frac{V(B_2^n)}{V(K)}\right)^{1/n} \\ &= nV(K)^{\frac{n-1}{n}} V(B_2^n)^{1/n}. \end{aligned} \tag{14}$$

For an n-dimensional unit ball, we have $S(B_2^n) = nV(B_2^n)$. Therefore,

$$\begin{aligned} \frac{S(K)}{S(B_2^n)} &\geq \frac{nV(K)^{\frac{n-1}{n}} V(B_2^n)^{1/n}}{} \\ \Rightarrow \left(\frac{S(K)}{S(B_2^n)}\right)^{\frac{1}{n-1}} &\geq \left(\frac{nV(K)^{\frac{n-1}{n}} V(B_2^n)^{1/n}}{nV(B_2^n)}\right)^{\frac{1}{n-1}} \\ &= \left(\frac{V(K)}{V(B_2^n)}\right)^{1/n} \end{aligned} \tag{15}$$

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3.3 Grunbaum's Theorem

Given a high-dimensional convex body, we would like to pick a point x such that for any cut of the body by a hyperplane, the piece containing x is big. A reasonable choice for x is the centroid, i.e.

$$x = \frac{1}{\text{Vol}(K)} \int_{y \in K} y dy.$$

This choice guarantees to get at least half of the volume for any origin symmetric body, such as a cube or a ball. The question is how much we are guaranteed to get for a general convex body, and in particular, what body gives the worst case. Do we get a constant fraction of the body, or does the guarantee depend on dimension?

Let us first consider the simple example of a circular n-dimensional cone (figure 3). Suppose we cut the cone C by the hyperplane $\{x_1 = \bar{x}_1\}$ at its centroid, where

$$\bar{x}_1 = \frac{1}{\mathrm{Vol}(C)} \int_{t=0}^{h} t \cdot \mathrm{Vol}_{n-1} \left(\frac{tR}{h}\right)^{n-1} dt = \frac{n}{n+1} h. \tag{16}$$

Grunbaum's theorem states that the circular cone is indeed the worst case if we choose the centroid.

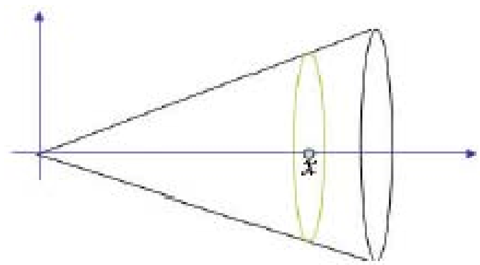

Figure 3: n-dimensional circular cone.

First we'll need the following lemma:

Lemma 5 *Let $L = C \cap \{x_1 \leq \bar{x}_1\}$ by the left side of the cone (which is x_1-aligned with vertex at the origin). Then $\frac{1}{2} \geq \frac{V(L)}{V(C)} \geq \frac{1}{e}$.*

Proof

$$\frac{V(L)}{V(C)} = \frac{V(\frac{n}{n+1}C)}{V(C)} = \left(\frac{n}{n+1}\right)^n$$

$$\frac{1}{2} \leq \left(\frac{n}{n+1}\right)^n \leq \frac{1}{e}$$

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Theorem 6 *(Grunbaum's Theorem) Let K be a convex body, and divide it into K_1 and K_2 using a hyperplane. If K_1 contains the centroid of K, then*

$$\frac{\mathrm{Vol}(K_1)}{\mathrm{Vol}(K)} \geq \frac{1}{e}. \tag{17}$$

In particular, the hyperplane through the centroid divides the volume into almost equal pieces, and the worst case ratio is approximately $0.37 : 0.63$.

Proof WLOG, change coordinates with an affine transformation so that the centroid is the origin and the hyperplane H used to cut is $x_1 = 0$. Then perform the following operations:

1. Replace every $(n-1)$-dimensional slice K_t with an $(n-1)$-dimensional ball with the same volume to get K', which is convex per Lemma 7 below.

2. Turn K' into a cone, such that the ratio gets smaller per Lemma 8 below.

Lemma 7 *K' is convex.*

Proof Let $K'_t = K' \cap \{x_1 = t\}$ be a parallel slice in the modified body. The radius of K'_t is proportional to $V(K_t)^{\frac{1}{n-1}}$. By applying Brunn-Minkowski inequality, we get that $V(K_t)^{\frac{1}{n-1}}$ is a concave function in t. Thus K' is convex. ■

Lemma 8 *We can turn K' into a cone while decreasing the ratio.*

Proof Let $K'_+ = K' \cap \{x_1 \geq 0\}, K'_- = K' \cap \{x_1 \leq 0\}$. Make a cone $y\bar{Q}_0$ by picking y having x_1 coordinate positive on the x_1-axis, and $V(y\bar{Q}_0) = V(K'_+$. Extend the code in the $\{x_1 \leq 0\}$ region, so that the volume of the extended part equals $V(K'_-)$; name this code C'. Now by Lemma 5, the centroid of C' must lie in $y\bar{Q}_0$. Let H' be the translation of H along the x_1-axis so that it contains the centroid of C'. Then

$$r(K, H) = r(C', H) \geq r(C', H') \geq 1/e.$$

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This completes the proof of Grunbaum's theorem. ■

4 Next Time

Next time, we will discuss approximating the volume of a convex body.