

Lecture 8

*Lecturer: Dan Spielman**Scribe: Jan Vondrák***Eigenvalues of planar graphs**

In this lecture, we prove the following theorem by Spielman and Shang-Hua Teng.

Theorem 1. *For any planar graph G on n vertices,*

$$\lambda_2(G) \leq \frac{8\Delta}{n}.$$

Also, we have Mihail's extension of Cheeger's inequality:

Theorem 2. *Let G be a graph of maximum degree Δ , L its Laplacian matrix, and v a vector orthogonal to $\mathbf{j} = (1, 1, \dots, 1)$. Then*

$$\Phi(G) \leq \sqrt{2\Delta \frac{v^T L v}{v^T v}}.$$

(A cut with this ratio can be found by ordering the vertices by the components of v .)

Combined with the bound on λ_2 for planar graphs, this yields a cut of ratio

$$\Phi \leq \sqrt{2\Delta \lambda_2} \leq \frac{4\Delta}{\sqrt{n}}$$

for planar graphs.

In the following, we use a generalized characterization of λ_2 .

Lemma 3 (Embedding lemma). *For any dimension $d \geq 1$,*

$$\lambda_2(G) = \min \left\{ \frac{\sum_{(i,j) \in E} \|x_i - x_j\|^2}{\sum_i \|x_i\|^2} : x_1, \dots, x_n \in \mathbb{R}^d, \sum_{i=1}^n x_i = \mathbf{0} \right\}.$$

Proof. For $d = 1$, this is the standard characterization

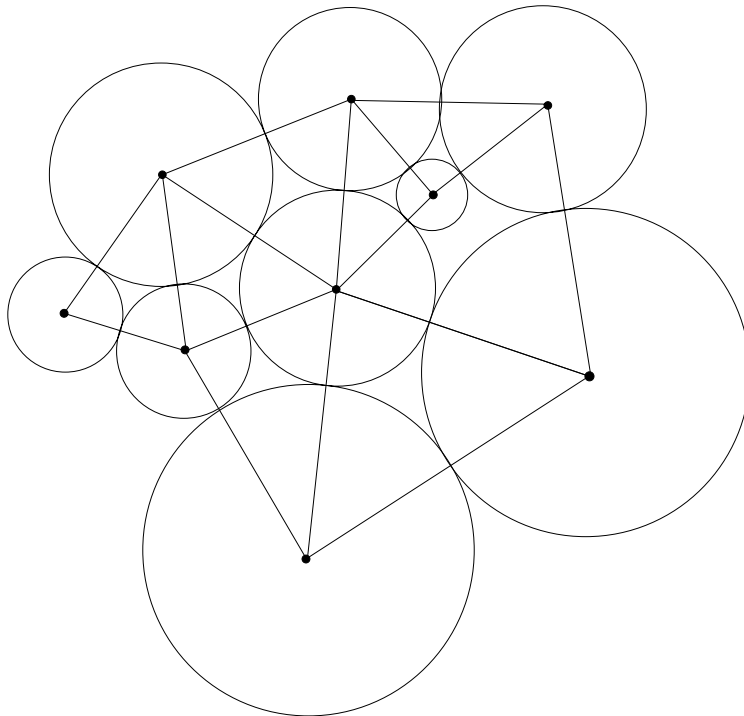
$$\lambda_2(G) = \min_{\sum x_i = 0} \frac{\sum_{(i,j) \in E} (x_i - x_j)^2}{\sum_i x_i^2}.$$

For $d \geq 1$, we can obviously achieve the same value by using only one dimension. However, we don't gain anything either, because

$$\frac{\sum_{(i,j) \in E} \|x_i - x_j\|^2}{\sum_i \|x_i\|^2} = \frac{\sum_{k=1}^d \sum_{(i,j) \in E} (x_{ik} - x_{jk})^2}{\sum_{k=1}^d \sum_i x_{ik}^2} \geq \min_k \frac{\sum_{(i,j) \in E} (x_{ik} - x_{jk})^2}{\sum_i x_{ik}^2}.$$

I.e., the minimum is achieved in one of dimensions as well. \square

Now we prove the bound for planar graphs by finding a suitable embedding of G in R^3 . Our starting point is Koebe's theorem about "kissing disks".

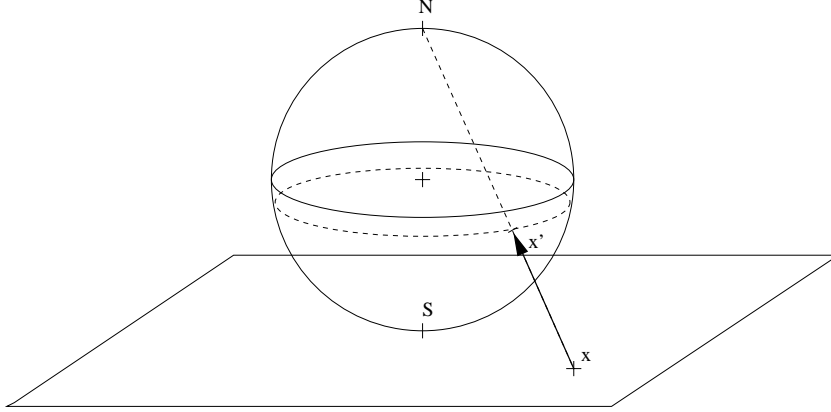


Theorem 4. For any planar graph G on n vertices, there is a set of interior-disjoint disks $\{D_1, D_2, \dots, D_n\}$ in the plane, such that

$$(i, j) \in E \iff D_i \cap D_j \neq \emptyset.$$

Using this representation as is (in R^2) wouldn't give a meaningful bound on λ_2 . Our approach is to lift it to a unit sphere in R^3 . If we obtain vectors such that $\sum_i x_i = 0$, we can use the embedding lemma to bound λ_2 . To lift the embedding, we apply the *stereographic projection*.

Definition 5 (Stereographic projection). Consider a plane and a unit sphere touching the plane in its origin. Call this point on the sphere the south pole, and the point on the sphere farthest away from the plane the north pole. For a point $x \in \mathbb{R}^2$, consider the line passing through x and the north pole. Define $\pi(x)$ to be the other point x' where this line intersects the sphere.



Note: This mapping is defined uniquely for each point in the plane. Conversely, each point on the sphere is the image of exactly one point, except the north pole. We can define the north pole to be the image of *infinity* if we want to have a bijection.

Another property of the stereographic projection is that circles in the plane are mapped onto circles on the sphere. (However, the center of a circle is not necessarily mapped onto the center of the corresponding circle on the sphere.) Thus we can use the projection to lift our disk embedding $\{D_1, \dots, D_n\}$ to obtain a disk embedding $\{\pi(D_1), \dots, \pi(D_n)\}$ on the sphere. It still holds that $\pi(D_i) \cap \pi(D_j) \neq \emptyset$ if and only if (i, j) is an edge.

Let x_i be the center of $\pi(D_i)$ on the sphere. It remains to ensure that $\sum x_i = 0$ but assume for now that this is true. We have

$$\|x_i\| = 1, \quad \sum_{i=1}^n \|x_i\|^2 = n.$$

Let r_i be the radius of the cap $\pi(D_i)$, measured in a straight line from x_i to the boundary of $\pi(D_i)$. If $(i, j) \in E$, the two caps touch, and therefore

$$\|x_i - x_j\|^2 \leq (r_i + r_j)^2 \leq 2(r_i^2 + r_j^2).$$

On the other hand, the caps are interior-disjoint and the sum of their areas does not exceed

the area of the sphere, therefore $\sum \pi r_i^2 \leq 4\pi$.

$$\sum_{(i,j) \in E} \|x_i - x_j\|^2 \leq 2 \sum_{(i,j) \in E} (r_i^2 + r_j^2) \leq 2 \sum_i d_i r_i^2 \leq 8\Delta.$$

For the second eigenvalue, we get

$$\lambda_2(G) \leq \frac{\sum_{(i,j) \in E} \|x_i - x_j\|^2}{\sum_i \|x_i\|^2} \leq \frac{8\Delta}{n}.$$

Finally, we show how to satisfy the condition $\sum x_i = 0$. We have some freedom in our choice of the projection π . First, we can expand the plane embedding by a constant factor. We can also choose where the plane touches the unit sphere.

Define $D^\beta : R^2 \rightarrow R^2$ as an expansion in the plane by a factor $\beta > 0$:

$$D^\beta(x) = \beta x.$$

Define $\pi_z : R^2 \rightarrow S^2$ as the stereographic projection such that the origin of the plane coincides with point z on the sphere.

Our goal is to use Brouwer's fixed point theorem.

Theorem 6. *Let B be the unit ball in R^3 and $\Phi : B \rightarrow B$ any continuous function. Then*

$$\exists x \in B; \quad \Phi(x) = x.$$

Corollary 7. *If $\Phi : B \rightarrow B$ is continuous and for any $x \in S^2$, $\Phi(x) = x$, then*

$$\exists x \in B; \quad \Phi(x) = 0.$$

This follows easily from Brouwer's theorem, because if there is a continuous function Φ which is identity on S^2 and doesn't map anything to 0, then

$$\Psi(x) = -\frac{\Phi(x)}{\|\Phi(x)\|}$$

would be a continuous function without any fixed point.

Now we define a mapping $f_\alpha : S^2 \rightarrow S^2$, for any $\alpha \in B$. For $\alpha = 0$, we define

$$f_0(z) = z.$$

For $0 < \|\alpha\| < 1$, we define

$$f_\alpha(z) = \pi_{\alpha/\|\alpha\|}(D^{1-\|\alpha\|}(\pi_{\alpha/\|\alpha\|}^{-1}(z))).$$

Finally, for $\|\alpha\| = 1$,

$$f_\alpha(-\alpha) = -\alpha$$

and

$$f_\alpha(z) = \alpha$$

for $z \neq \alpha$.

Let the centers of caps $\pi(D_1), \dots, \pi(D_n)$ be x_1, \dots, x_n . Note that for a fixed x_i , $f_\alpha(x_i)$ changes continuously depending on α , unless α approaches the antipode $-x_i$ on the boundary of B . To avoid this singularity, we define

$$\omega_i(\alpha) = \min\left\{\frac{2 - \|\alpha - x_i\|}{\epsilon}, 1\right\}$$

and

$$\Phi(\alpha) = \frac{\sum_{i=1}^n \omega_i(\alpha) f_\alpha(x_i)}{\sum_{i=1}^n \omega_i(\alpha)}.$$

Then $\Phi : B \rightarrow B$ is continuous, because whenever α approaches the antipode of some x_i , the factor $\omega_i(\alpha)$ annihilates the influence of $f_\alpha(x_i)$. By the corollary, there exists $y \in B$ such that $\Phi(y) = 0$. It remains to choose $\epsilon > 0$ small enough so that $\|y - x_i\| < 2 - \epsilon$ for all i . This is possible, because for y very close to the boundary of B , $f_y(x)$ maps everything very close to y and therefore $\Phi(y)$ cannot be zero unless y is bounded away from the boundary (by some constant depending on the configuration of the x_i 's). It follows that $\forall i; \omega_i(y) = 1$ and

$$\Phi(y) = \frac{1}{n} \sum_{i=1}^n f_y(x_i) = 0.$$

Therefore $\{f_y(x_1), \dots, f_y(x_n)\}$ is our "centered" embedding for which the embedding lemma can be applied.